Article:

Bowman-Scargill, Chris orcid.org/0000-0001-6046-8930 (2022) The many integral graded cellular bases of hecke algebras of complex reflection groups. American Journal of Mathematics. pp. 437-504. ISSN 0002-9327

https://doi.org/10.1353/ajm.2022.0008

THE MANY INTEGRAL GRADED CELLULAR BASES OF HECKE ALGEBRAS OF COMPLEX REFLECTION GROUPS

C. BOWMAN

We settle several long-standing problems in the theory of cyclotomic Hecke algebras: for each charge we construct the integral cellular basis predicted by Ariki's categorification theorem. We hence prove unitriangularity of decomposition matrices and Martin–Woodcock's conjecture.

Introduction

There are two remarkably successful approaches to the study of Hecke algebras of symmetric groups: the first is via geometry and the second is via categorical Lie theory. The Kazhdan–Lusztig basis has deep *geometric* origins (arising as the shadow of an intersection cohomology sheaf on a variety); this basis enjoys many positivity properties, however it is inhomogenous with respect to the Hecke algebra's graded structure. The graded Murphy basis arises in *categorical Lie theory*, it encodes the graded induction and restriction along the tower of Hecke algebras, and it is simpler and more explicit. The most important property shared by the Kazhdan–Lusztig and graded Murphy bases is that they are both integral cellular bases [KL79, HM10].

The complex reflection groups were classified into the infinite series $G(\ell, d, n)$ and 34 exceptional cases by Shephard–Todd [ST54]; their corresponding Hecke algebras were later defined by Ariki and Koike [AK94, Ari95], for the infinite families, and Broué–Malle–Rouquier [BMR98], in general. For every *real reflection group*, Lusztig has constructed *many different* Kazhdan–Lusztig bases for the associated Hecke algebras [Lus83, Lus03]. However, this is as far as the geometric picture (and the underlying Kazhdan–Lusztig bases!) can be pushed: *there do not exist Kazhdan–Lusztig bases for complex reflection groups or their Hecke algebras.*

Categorical Lie theory picks up where geometry leaves off (one of the most spectacular examples to-date being [EW14]). In particular, while complex reflection groups *do not* possess Kazhdan–Lusztig bases, Ariki's categorification theorem suggests that every choice of charge should give rise to a corresponding cellular structure on the Hecke algebra of type $G(\ell, 1, n)$ [Ari02]. We prove that every charge does indeed give rise to an integral cellular basis on the Hecke algebra of type $G(\ell, 1, n)$, as has long been hoped and expected. Namely we generalise the graded Murphy bases from asymptotic charges [HM10] to *all possible charges* on all Hecke algebras of type $G(\ell, 1, n)$. (Corresponding bases for type $G(\ell, d, n)$ can be constructed from ours via Clifford theory [HMR].)

In order to state our main result, we first require some notation. For the purposes of the introduction, we let $\Bbbk$ be a field. Given $\sigma = (e; \sigma_0, \sigma_1, \ldots, \sigma_{\ell-1}) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$, we define the cyclotomic Hecke algebra to be the $\Bbbk$-algebra generated by $T_0, T_1, \ldots T_{n-1}$ subject to the relations

$$(T_i + q)(T_i - 1) = 0 \quad (T_0 - q^{\sigma_0})(T_0 - q^{\sigma_1}) \ldots (T_0 - q^{\sigma_{\ell-1}}) = 0$$
$$T_iT_j = T_jT_i \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad T_0T_1T_0T_1 = T_1T_0T_1T_0$$

for q an eth root of unity and $1 \leqslant i, j < n$, $|i - j| > 1$. The starting point for this paper is the observation that this presentation depends only on the reduction of σ modulo e (which we denote by $\underline{s} \in \mathbb{N}_{>1} \times (\mathbb{Z}/e\mathbb{Z})^\ell$). We denote the cyclotomic Hecke algebra by $H_n^{\Bbbk}(\underline{s})$ in order to emphasise the independence of the actual charge. For each distinct integral lift $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ of $\underline{s} \in \mathbb{N}_{>1} \times (\mathbb{Z}/e\mathbb{Z})^\ell$, we have a corresponding $\mathbf{a}_\sigma$-order on $\mathscr{P}_n^\ell$ (the ℓ-multipartitions of n) due to Lusztig and an $\mathbf{a}_\sigma$-grading on standard tableaux due to Uglov.

Theorem A. *The algebra $H_n^{\Bbbk}(\underline{s})$ has many graded cellular structures, one for each integral lift $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$. The basis*

$$\{A^\sigma_{\mathsf{st}} \mid \lambda \in \mathscr{P}_n^\ell, \mathsf{s}, \mathsf{t} \in \mathrm{Std}_\sigma(\lambda)\}$$

is cellular with respect to Lusztig's $\mathbf{a}_\sigma$-order on $\mathscr{P}_n^\ell$ and Uglov's $\mathbf{a}_\sigma$-grading on standard tableaux.

The algebra $H_n^{\Bbbk}(\underline{s})$ has many (non-isomorphic) quasi-hereditary covers (one for each integral lift $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$) and we shall see (in Theorem 6.23) that each of the bases of Theorem A arises by idempotent truncation of a cellular basis of the corresponding quasi-hereditary cover. (In exactly the same manner that Murphy's basis of $\Bbbk\mathfrak{S}_n$ is obtained from Green's co-determinant basis of the Schur algebra.)

Theorem A allows us prove that the decomposition matrices are unitriangular with respect to all Lusztig $\mathbf{a}_\sigma$-orderings on $\mathscr{P}_n^\ell$ over any field and explicitly construct the irreducible modules parameterised by Ariki's categorification theorem. This completes a long history of work on this topic [Ari01, Ari96, AM00, BI03, BGIL10, BJ09, CJ11, CJ12, CJ16, CGG12, DJM95, DJM98, Gec98, Gec07b, CJ16, CGG12, GJ11, GM09, GR01, GJ06, Jac04, Jac05, Jac07, Jac11].

Theorem B. *Let $\Bbbk$ be a field. For each integral lift $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$, the irreducible $H_n^{\Bbbk}(\underline{s})$-modules are explicitly constructible as canonical quotients of the cell modules $\mathrm{S}_\sigma^{\Bbbk}(\lambda)$ labelled by the associated set of Uglov ℓ-partitions Σ_n^ℓ (see Section 10 for definition) and the decomposition matrix is unitriangular with respect to Lusztig's $\mathbf{a}_\sigma$-ordering on $\mathscr{P}_n^\ell$.*

It is worth stressing that there *do not exist* Kazhdan–Lusztig bases for complex reflection groups (in particular for type $G(\ell,1,n)$ for $\ell > 2$). The Spets programme seeks to generalise the Kazhdan–Lusztig theory, existence of finite groups of Lie type, and the structural properties of Hecke algebras from Weyl groups to the wider family of complex reflection groups. Our integral cellular bases generalise one tranche of this theory (the strong structural properties of Hecke algebras which normally depend on the existence of Kazhdan–Lusztig bases) to type $G(\ell,1,n)$.

Each of the integral graded cellular bases we construct provides us with a new viewpoint from which to study the Hecke algebra: a new family of Specht modules, a new filtration on the projective modules (this was Geck–Rouquier's motivation for instigating this research programme in [Gec98, GR01]), a new grading and new unitriangular ordering on the decomposition matrix, and most importantly *a new $\mathbb{Z}$-lattice on the Hecke algebra*. Therefore our many different integral cellular bases provide us with many new ways to study the *modular* representations of Hecke algebras by "reduction modulo p". Each of our new $\mathbb{Z}$-lattices gives us a new way of factorising representation theoretic questions (e.g. decomposition numbers) via a two step process: first calculate the decomposition numbers of the Hecke algebra over $\mathbb{Q}$ in terms of Kazhdan–Lusztig polynomials and then calculate the corresponding 'p-modular adjustment matrices'. All known results on Hecke algebras in positive characteristic have been proven within the framework of the *asymptotic* cellular structure of [DJM98, HM10] (e.g. the Jantzen sum formula [JM00], homological structure [LM07, LM14, LM10, FS16], branching rules [Ari06], and decomposition numbers [RW, EL]). We vastly generalise this framework from *asymptotic* charges to *all* weightings and hence prove:

Theorem C (Martin–Woodcock's conjecture)**.** *There is a square submatrix of the decomposition matrix of $H_n^{\mathbb{Q}}(\underline{s})$ with entries given by the non-parabolic Kazhdan–Lusztig polynomials of type $\widehat{A}_{\ell-1}$.*

Finally, we generalise all the results of Brundan–Kleshchev–Wang [BKW11] to arbitrary weightings; in particular the graded branching rule. Fix a weighting $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ and the corresponding sets of Specht and irreducible modules $\{\mathrm{S}_\sigma^{\Bbbk}(\lambda) \mid \lambda \in \mathscr{P}_n^\ell\}$ and $\{L_\sigma^{\Bbbk}(\lambda) \mid \lambda \in \Sigma_n^\ell \subseteq \mathscr{P}_n^\ell\}$. We would like to understand the structure of the restrictions of these modules to the subalgebra $H_{n-1}^{\Bbbk}(\underline{s}) \subset H_n^{\Bbbk}(\underline{s})$ (see also [FLOTW99, Ari96, Ari06, AM00, BKW11, Mat18]).

Theorem D. *Let $\lambda \in \mathscr{P}_n^\ell$ and let $\alpha_1 \rhd_\sigma \alpha_2 \rhd_\sigma \cdots \rhd_\sigma \alpha_z$ denote the removable boxes of λ. Then the restriction of $\mathrm{S}_\sigma^{\Bbbk}(\lambda)$ has an $H_{n-1}^{\Bbbk}(\underline{s})$-module filtration*

$$0 = \mathrm{S}_\sigma^{z+1,\lambda} \subset \mathrm{S}_\sigma^{z,\lambda} \subset \cdots \subset \mathrm{S}_\sigma^{1,\lambda} = \mathrm{Res}_{H_{n-1}^{\Bbbk}(\underline{s})}(\mathrm{S}_\sigma^{\Bbbk}(\lambda))$$

such that for each $1 \leqslant r \leqslant z$, we have $\mathrm{S}_\sigma^{\Bbbk}(\lambda - \alpha_r)\langle \deg(\alpha_r)\rangle \cong \mathrm{S}_\sigma^{r,\lambda}/\mathrm{S}_\sigma^{r+1,\lambda}$.

Antecedents: We re-iterate that there are many different Kazhdan–Lusztig bases on a given Hecke algebra of a real reflection group, one for each choice of weighting [Lus03, Lus83]; the "canonical" basis of [KL79] is then obtained by restricting ones attention to the trivial weighting. These many weightings have applications in Schubert varieties [Lus83] statistical mechanics [MS94, MW03] and provide many different lenses through which to view and understand a given Hecke algebra.

- Canonical basic sets and cellularity: The search for a proof of Theorems A and B was the principal focus of a book by Geck–Jacon [GJ11] and a multitude of conjectures [DJM95, BGIL10, BJ09, AJ10] as well as being one of the motivating factors for the recent surge of interest in Cherednik algebras [CGG12, GM09, BR12, BR13] and [GGOR03, Section 6]. Over a field of characteristic zero, a huge literature has focussed on constructing the combinatorial shadows of our bases in Theorem A; these shadows are called "canonical basic sets" and were first introduced by Geck–Rouquier [GR97, GR01]. These combinatorial shadows have been intensely studied [BI03, BGIL10, BJ09, CJ11, CJ12, CJ16, CGG12, GIP08, GI13, Gec98, Gec07b, GJ11, GM09, GR01, GJ06, Jac04, Jac05, Jac07, Jac11] and have been used to prove unitriangularity of decomposition matrices with respect to the Lusztig $\mathbf{a}_\sigma$-orderings over $\mathbb{Q}$. Our Theorems A and B lift these results to a higher structural level and extend them to arbitrary fields.

  In the case of asymptotic charges (for which $\sigma_i \gg \sigma_{i+1}$ for $0 \leqslant i < \ell-1$) the combinatorics and basis of Theorem A coincides with that of [HM10, Main Theorem] and Theorem A generalises the main results of [HM10] to all possible charges. The existing results on cellular bases of Hecke algebras of type $G(\ell,1,n)$ form along two axes: for $\ell \in \{1,2\}$ cellular Kazhdan–Lusztig theoretic bases exist for all charges [Lus03, Lus83, Gec07a]; for asymptotic charges cellular Murphy-type bases exist for all types $G(\ell,1,n)$ [HM10]. This paper completes the cellularity picture *along both these axes* by constructing cellular bases for all charges on all cyclotomic Hecke algebras.
- Parameterising and constructing irreducible modules: Ariki's categorification theorem gives rise to many abstract parameterisations of irreducible $H_n^{\Bbbk}(\underline{s})$-modules [Ari02]. The aforementioned asymptotic cellular structure of [HM10] is the key ingredient in the explicit construction of irreducible modules as canonical quotients of Specht modules labelled by *Kleshchev* ℓ-partitions in [Ari01, AM00]. However, the Kleshchev ℓ-partitions provide just one of many possible labellings of the nodes in the crystal graph [CGG12]; each such labelling should give rise to an explicit construction of the irreducible modules. In Section 10, we provide these many different constructions of the irreducible modules (one for each possible charge) and over *arbitrary fields*. For each charge, we shall see that the corresponding irreducibles are those which survive under the associated KZ functor.

This paper has gone through many iterations over the past few years, we have provided a discussion of how to pass between these versions at the end of the current paper. In particular, earlier versions of this paper did not use Theorem A (in conjunction with results of Jacon [Jac07]) in order to deduce that the Uglov multipartitions label the irreducible modules of cyclotomic Hecke algebras (see Theorem B). Using our results, Kerschl has independently used our Theorem A in order to deduce this labelling result [Ker]. Our proof is simpler (as it makes use of earlier results [Jac07]) but Kerschl's proof has the added advantage of providing new lower bounds for the dimensions of these irreducible modules.

1. Weighted combinatorics of complex reflection groups

For the remainder of the paper, unless otherwise specified, let $\Bbbk$ be an arbitrary integral domain.

We let $\mathfrak{S}_n$ denote the symmetric group with the usual Coxeter generators $s_{i,i+1}$ for $1 \leqslant i < n$. Given parameters q and $(Q_0, Q_1, \dots, Q_{\ell-1})$ we define the Hecke algebra of $(\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n$ to be the $\Bbbk$-algebra $H_n^{\Bbbk}(q; Q_0, \dots, Q_{\ell-1})$ generated by $T_0, T_1, \dots T_{n-1}$ subject to the relations

$$\begin{gathered} (T_i + q)(T_i - 1) = 0 \quad (T_0 - Q_0)(T_0 - Q_1)\dots(T_0 - Q_{\ell-1}) = 0 \\ T_iT_j = T_jT_i \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad T_0T_1T_0T_1 = T_1T_0T_1T_0 \end{gathered} \tag{1.1}$$

for $1 \leqslant i,j < n$ and $|i-j| > 1$. We set $X_j = q^{1-j}T_{j-1}\dots T_1T_0T_1\dots T_{j-1}$. Given a charge $\sigma = (e; \sigma_0, \sigma_1, \dots, \sigma_{\ell-1}) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ we are interested in the specialisation of the parameters $q = \xi$ a primitive eth root of unity and $Q_m = q^{\sigma_m}$ for $0 \leqslant m < \ell$. Given $\sigma = (e; \sigma_0, \sigma_1, \dots, \sigma_{\ell-1}) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$, we define the e-charge to be $\underline{s} = (e; s_0, s_1, \dots, s_{\ell-1}) \in \mathbb{N}_{>1} \times (\mathbb{Z}/e\mathbb{Z})^\ell$ obtained by reducing the ℓ-tuple modulo e. After specialisation we obtain the algebra $H_n^{\Bbbk}(\underline{s}) := H_n^{\Bbbk}(\xi; \xi^{\sigma_0}, \dots, \xi^{\sigma_{\ell-1}})$ which we defined in the introduction; the notation has been chosen to emphasise that, after specialisation, the definition of the Hecke algebra depends only on the e-charge.

1.1. **Charged ℓ-partitions.** Fix a charge $\sigma = (e; \sigma_0, \dots, \sigma_{\ell-1}) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$. We define a configuration of boxes to be a subset of

$$\{(r, c, m) \mid r, c, m \in \mathbb{N}, 1 \leqslant r, c \leqslant n, 0 \leqslant m < \ell\} \tag{$\square$}$$

and we let $\mathscr{C}_n^\ell$ denote the set of all configurations of n boxes. We refer to a box (r, c, m) as being in the rth row and cth column of the mth component of the configuration. Given a box, (r, c, m), we define the content of this box to be $\mathsf{ct}(r, c, m) = \sigma_m + c - r$ and we define its residue to be $\mathrm{res}(r, c, m) \equiv \mathsf{ct}(r, c, m) \pmod e$. We refer to a box of residue $i \in \mathbb{Z}/e\mathbb{Z}$ as an i-box.

We define a partition, λ, of n to be a finite weakly decreasing sequence of non-negative integers $(\lambda_1, \lambda_2, \dots)$ whose sum, $|\lambda| = \lambda_1 + \lambda_2 + \dots$, equals n. An ℓ-partition $\lambda = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ of n is an ℓ-tuple of partitions such that $|\lambda^{(0)}| + \dots + |\lambda^{(\ell-1)}| = n$. We denote the set of ℓ-partitions of n by $\mathscr{P}_n^\ell$. Given $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell-1)}) \in \mathscr{P}_n^\ell$, the Young diagram is the configuration of boxes,

$$[\lambda] = \{(r, c, m) \mid 1 \leqslant c \leqslant \lambda_r^{(m)}\}.$$

We now recall Lusztig's $\mathbf{a}_\sigma$-ordering on ℓ-partitions and Webster's coarsening of this ordering.

Definition 1.1. Given $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ a charge, we write $(r, c, m) <_\sigma (r', c', m')$ if either

(*i*) $\mathsf{ct}(r, c, m) < \mathsf{ct}(r', c', m')$ or
(*ii*) $\mathsf{ct}(r, c, m) = \mathsf{ct}(r', c', m')$ and $m > m'$

We write $(r, c, m) \lhd_\sigma (r', c', m')$ if both $(r, c, m) <_\sigma (r', c', m')$ and $\mathsf{res}(r, c, m) = \mathsf{res}(r', c', m')$.

The following formulation of the Lusztig $\mathbf{a}_\sigma$-ordering is given in [CGG12, 5.6 Proposition].

Definition 1.2 (Lusztig's $\mathbf{a}_\sigma$-ordering)**.** For $\lambda, \mu \in \mathscr{P}_n^\ell$, we write $\mu \leqslant_\sigma \lambda$ if there is a bijective map $\mathsf{A} : [\lambda] \to [\mu]$ such that either $\mathsf{A}(r, c, m) <_\sigma (r, c, m)$ or $\mathsf{A}(r, c, m) = (r, c, m)$ for all $(r, c, m) \in \lambda$.

We now rephrase Webster's ordering on $\mathscr{P}_n^\ell$ in such a way that it is easily seen to be a coarsening of Lusztig's $\mathbf{a}_\sigma$-ordering. We reconcile this with Webster's original diagrammatic definition shortly.

Definition 1.3 (Webster's ordering)**.** For $\lambda, \mu \in \mathscr{C}_n^\ell$, we write $\mu \trianglelefteq_\sigma \lambda$ if there is a residue preserving bijective map $\mathsf{A} : [\lambda] \to [\mu]$ such that either $\mathsf{A}(r, c, m) \lhd_\sigma (r, c, m)$ or $\mathsf{A}(r, c, m) = (r, c, m)$ for all $(r, c, m) \in \lambda$.

We now discuss how Definition 1.1 and the ensuing orderings on $\mathscr{P}_n^\ell$ can be visualised diagrammatically. Given $\lambda \in \mathscr{P}_n^\ell$, the associated (mirrored) σ-Russian array is defined as follows. (We drop the prefix "mirrored" for the remainder of this paper, we just highlight now for the reader that our conventions are the opposite of the usual definition of a Russian diagram.) For each $0 \leqslant m < \ell$, we place a point on the real line at $\sigma_m - \frac{m}{\ell}$ and consider the region bounded by half-lines at angles $3\pi/4$ and $\pi/4$. (Compare the m/ℓ removed from the charge with condition (*ii*) of Definition 1.1.) We tile the resulting quadrant with a lattice of squares, each with diagonal of length 2 (this will be important!). We place the box $(1, 1, m)$ at the point $\sigma_m - \frac{m}{\ell}$ on the real line, with rows going northeast from this node, and columns going northwest. Given a fixed charge $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ and $\lambda \in \mathscr{P}_n^\ell$, we do not distinguish between the configuration of boxes and its σ-Russian array.

Proposition 1.4. *We have that $(r, c, m) <_\sigma (r', c', m')$ if and only if the box (r, c, m) appears strictly to the left of the box (r', c', m') in the σ-Russian array.*

Proof. This is clear from the definitions. Notice that the subtraction $-m/\ell$ ensures that (*ii*) of Definition 1.1 matches the diagrammatic ordering. $\square$

Example 1.5. A charge is said to be asymptotic if $\sigma_m - \sigma_{m+1} > n$ for all $0 \leqslant m < \ell - 1$. For $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ a asymptotic charge and $\lambda, \mu \in \mathscr{P}_n^\ell$, it is easy to see that if $\lambda \geqslant_\sigma \mu$ if and only if

$$\sum_{i=1}^{k-1} |\lambda^{(i)}| + \sum_{i=1}^{j} \lambda_i^{(k)} \geqslant \sum_{i=1}^{k-1} |\mu^{(i)}| + \sum_{i=1}^{j} \mu_i^{(k)}$$

for all $1 \leqslant k \leqslant \ell$ and $1 \leqslant j \leqslant n$.

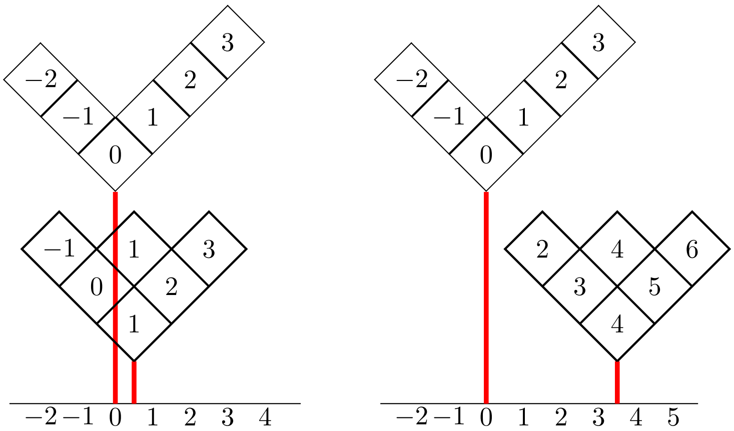


FIGURE 1. We picture the 2-partition $((4,1^2) \mid (3,2,1))$ for $(e;\sigma_0,\sigma_1) = (e;0,1)$ and $(e;\sigma_0,\sigma_1) = (e;0,4)$ respectively for $e \in \mathbb{N}_{>1}$. In each box we have placed the content of the box. Notice that the boxes of content 1 in the second component appear to the left of those of content 1 in the first component (by half a unit).

Example 1.6. In the case $\sigma = (e;\sigma_0,\sigma_1,\ldots,\sigma_{\ell-1}) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ is such that $0 < \sigma_i - \sigma_j < e$ for $0 \leqslant i < j < \ell$, the σ-dominance order coincides with the ordering on $\mathscr{P}_n^\ell$ considered in [FLOTW99]. This charge is considered in greater detail in Section 13.

1.2. **Charged standard tableaux.** Given $\lambda \in \mathscr{P}_n^\ell$, we let $\mathrm{Rem}(\lambda)$ (respectively $\mathrm{Add}(\lambda)$) denote the set of all removable (respectively addable) boxes of the Young diagram of λ so that the resulting diagram is the Young diagram of a ℓ-partition. We extend the residue and dominance notation above in the obvious fashion. Given $i \in \mathbb{Z}/e\mathbb{Z}$, we let $\mathrm{Rem}_i(\lambda) \subseteq \mathrm{Rem}(\lambda)$ and $\mathrm{Add}_i(\lambda) \subseteq \mathrm{Add}(\lambda)$ denote the subsets of boxes of residue $i \in \mathbb{Z}/e\mathbb{Z}$.

Definition 1.7. Fix $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$. Given $\lambda \in \mathscr{P}_n^\ell$, we define a σ-tableau of shape λ to be a bijective map from the boxes of the σ-Russian array of λ to the set $\{1,\ldots,n\}$ (depicted as a filling the boxes with the corresponding integers). We define a standard tableau to be a tableau in which the entries increase along the rows and columns of each component. We let $\mathrm{Std}_\sigma(\lambda)$ denote the set of all standard tableaux of shape $\lambda \in \mathscr{P}_n^\ell$. Given $\mathsf{t} \in \mathrm{Std}_\sigma(\lambda)$, we set $\mathrm{Shape}(\mathsf{t}) = \lambda$. Given $1 \leqslant k \leqslant n$, we let $\mathsf{t}{\downarrow_{\{1,\ldots,k\}}}$ be the subtableau of t whose entries belong to the set $\{1,\ldots,k\}$. For $\mathsf{s},\mathsf{t} \in \mathrm{Std}_\sigma(\lambda)$ we write $\mathsf{s} \trianglelefteq_\sigma \mathsf{t}$ if $\mathrm{Shape}(\mathsf{s}{\downarrow_{\{1,\ldots,k\}}}) \trianglelefteq_\sigma \mathrm{Shape}(\mathsf{t}{\downarrow_{\{1,\ldots,k\}}})$ for $1 \leqslant k \leqslant n$ (one can define $\leqslant_\sigma$ on $\mathrm{Std}_\sigma(\lambda)$ similarly).

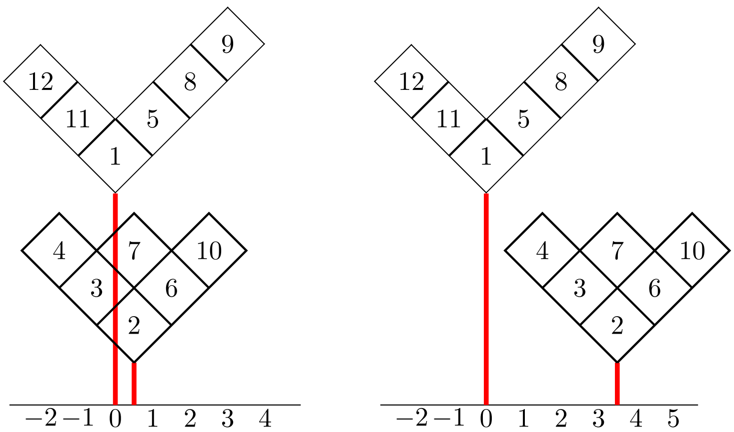


FIGURE 2. Tableaux $\mathsf{s} \in \mathrm{Std}_{(3;0,1)}(\lambda)$ and $\mathsf{t} \in \mathrm{Std}_{(3;0,4)}(\lambda)$ for $\lambda = ((4,1^2),(3,2,1))$.

Definition 1.8. We define a residue sequence to be an element $\imath = (i_1,\ldots,i_n) \in (\mathbb{Z}/e\mathbb{Z})^n$. Given $\mathsf{t} \in \mathrm{Std}_\sigma(\lambda)$ we define the residue sequence, $\imath_\mathsf{t}$, as follows,

$$\mathrm{res}(\mathsf{t}) = (\mathrm{res}(\mathsf{t}^{-1}(1)), \mathrm{res}(\mathsf{t}^{-1}(2)), \ldots, \mathrm{res}(\mathsf{t}^{-1}(n))) \in (\mathbb{Z}/e\mathbb{Z})^n.$$

Example 1.9. The algebras $H^{\Bbbk}_{12}(3;0,1)$ and $H^{\Bbbk}_{12}(3;0,4)$ are isomorphic. For $\lambda = ((4,1^2),(3,2,1))$ we have pictured a tableau $\mathsf{s} \in \mathrm{Std}_{(3;0,1)}(\lambda)$ and $\mathsf{t} \in \mathrm{Std}_{(3;0,4)}(\lambda)$ in Figure 2. The residue sequences s and t from Figure 2 are all the same and are equal to $(0,1,0,2,1,2,1,2,0,0,2,1)$.

Definition 1.10. Let $\lambda \in \mathscr{P}^\ell_n$ and $\mathsf{t} \in \mathrm{Std}_\sigma(\lambda)$. We let $\mathsf{t}^{-1}(k)$ denote the box in t containing the integer $1 \leqslant k \leqslant n$. Given $1 \leqslant k \leqslant n$, we let $\mathcal{A}_\mathsf{t}(k)$, (respectively $\mathcal{R}_\mathsf{t}(k)$) denote the set of all addable $\mathsf{res}(\mathsf{t}^{-1}(k))$-boxes (respectively all removable $\mathsf{res}(\mathsf{t}^{-1}(k))$-boxes) of the ℓ-partition $\mathrm{Shape}(\mathsf{t}{\downarrow}_{\{1,\dots,k\}})$ which are less than $\mathsf{t}^{-1}(k)$ in the σ-dominance order (i.e those which appear to the *left* of $\mathsf{t}^{-1}(k)$).

Definition 1.11. Let $\lambda \in \mathscr{P}^\ell_n$ and $\mathsf{t} \in \mathrm{Std}_\sigma(\lambda)$. We define the degree of t as follows,

$$\deg(\mathsf{t}) = \sum_{k=1}^{n} \left(|\mathcal{A}_\mathsf{t}(k)| - |\mathcal{R}_\mathsf{t}(k)|\right).$$

Remark 1.12. For $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ an asymptotic charge, our tableaux and grading coincide with those of [HM10, Section 3] and [BKW11, Section 1].

Example 1.13. We continue with the example above specialising $e = 3$, and $\sigma = (3;0,1)$ versus $\sigma = (3;0,4)$. The tableau s of Figure 2 has degree 5: the boxes with entries 5, 7, 8, 9, 10 and 11 have degrees $1,1,1,2,1,-1$ respectively and all other boxes have degree 0. The tableau t of Figure 2 has degree 0: the boxes with entries 2, 3, 4, 9 have degrees $1,-1,1,-1$ respectively and all other boxes have degree 0. We note that the boxes of t with entries 6, 7, 8, 10 all have degree zero because they have both an addable and a removable node to their left which cancel out.

1.3. **Charged semistandard tableaux.** We first tilt the σ-Russian array of $\lambda \in \mathscr{C}^\ell_n$ ever-so-slightly in the anticlockwise direction so that the top vertex of the box (r,c,m) has x-coordinate

$$\mathbf{I}^\sigma_{(r,c,m)} = \mathsf{ct}(r,c,m) - m/\ell - (r+c)\varepsilon$$

for $\varepsilon \ll \frac{1}{2n\ell}$ (up to small angle approximation). Our assumption that $\varepsilon \ll \frac{1}{2n\ell}$ implies that no two boxes in the σ-charged Young diagram of $\lambda \in \mathscr{C}^\ell_n$ can have the same x-coordinate and thus we have refined the ordering $\leqslant_\sigma$ of Definition 1.1 to a total ordering on boxes. Given $\lambda \in \mathscr{C}^\ell_n$, we let $\mathbf{I}^\sigma_\lambda$ denote the ordered set of the $\mathbf{I}^\sigma_{(r,c,m)}$ for $(r,c,m) \in \lambda$. Given $\lambda \in \mathscr{C}^\ell_n$, the associated $\mathsf{residue\ sequence}$, $\mathrm{res}(\lambda)$, of λ is given by reading the residues of the boxes of λ according to the natural ordering on x-coordinates.

Definition 1.14. Given $\lambda, \mu \in \mathscr{C}^\ell_n$ we define a tableau, T, of shape λ and weight μ to be a bijective map $\mathsf{T} : [\lambda] \to \mathbf{I}^\sigma_\mu$. We say that a tableau is $\mathsf{semistandard}$ if it also satisfies the following properties

(i) $\mathsf{T}(1,1,m) < \sigma_m$,
(ii) $\mathsf{T}(r,c,m) < \mathsf{T}(r-1,c,m) - 1$,
(iii) $\mathsf{T}(r,c,m) < \mathsf{T}(r,c-1,m) + 1$,

for $(r,c,m) \in \lambda$. We denote the set of all semistandard tableaux of shape λ and weight μ by $\mathrm{SStd}_\sigma(\lambda,\mu)$. Given $\mathsf{T} \in \mathrm{SStd}_\sigma(\lambda,\mu)$, we write $\mathrm{Shape}(\mathsf{T}) = \lambda$.

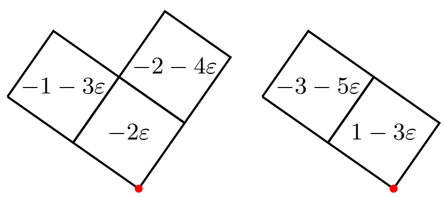


FIGURE 3. A semistandard tableau $\mathsf{S} \in \mathrm{SStd}_{(3;4,0)}(((1^2),(2,1)),(\varnothing,(2,1^3)))$. We have tilted the components of this 2-partition ε units anti-clockwise so that the x-coordinate of a box (r,c,m) is equal to $\mathsf{ct}(r,c,m) - m/\ell - (r+c)\varepsilon$. Two possible corresponding basis elements for this tableau are depicted in Figure 15.

2. Graded cellular algebras and canonical basic sets

Let R be an integral domain with field of fractions $\mathbb{F}$. Let A^R be an associative R-algebra which is finitely generated and free over R and $\vartheta : R \to \mathbb{K}$ a ring homomorphism into a field $\mathbb{K}$ such that $\mathbb{K}$ is the field of fractions of $\vartheta(R)$. We obtain an $\mathbb{F}$-algebra $A^{\mathbb{F}} = A^R \otimes_R \mathbb{F}$ and a $\mathbb{K}$-algebra $A^{\mathbb{K}} = A^R \otimes_R \mathbb{K}$. We let $\mathrm{Irr}(A^{\mathbb{K}})$ (respectively $\mathrm{Irr}(A^{\mathbb{F}})$) denote the set of all irreducible representations of $A^{\mathbb{K}}$ (respectively $A^{\mathbb{F}}$) up to isomorphism. The following generalises the definition of [GJ11, Section 3.1.7] to more general modular systems.

Definition 2.1. Let A^R be an algebra with representations $\{V_\lambda^R \mid \lambda \in \Pi\}$. Suppose that $A^{\mathbb{F}}$ is semisimple and $\{V_\lambda^{\mathbb{F}} := V_\lambda^R \otimes_R \mathbb{F} \mid \lambda \in \Pi\} = \mathrm{Irr}(A^{\mathbb{F}})$. Let $\rhd$ be a partial order on Π such that:

- (i) Given $L_\lambda^{\mathbb{K}} \in \mathrm{Irr}(A^{\mathbb{K}})$, let $\mathscr{S}_{\trianglerighteq}(L_\lambda^{\mathbb{K}}) = \{V_\mu^{\mathbb{F}} \mid \mu \in \Pi, V_\mu^R \otimes_R \mathbb{K}$ has $L_\lambda^{\mathbb{K}}$ as a composition factor$\}$. Then the set $\mathscr{S}_{\trianglerighteq}(L_\lambda^{\mathbb{K}})$ contains a unique minimal element, $V_\lambda^{\mathbb{F}}$, with respect to $\rhd$.
- (ii) There exists an injective map $\mathrm{Irr}(A^{\mathbb{K}}) \to \mathrm{Irr}(A^{\mathbb{F}})$.
- (iii) For all $L_\lambda^{\mathbb{K}} \in \mathrm{Irr}(A^{\mathbb{K}})$, we have that $L_\lambda^{\mathbb{K}}$ appears exactly once as a composition factor of $V_\lambda^R \otimes_R \mathbb{K}$.

If this holds, we say that $B_{\rhd}^{\mathbb{K}} = \{\lambda \mid L_\lambda \in \mathrm{Irr}(A^{\mathbb{K}})\}$ is a canonical basic set for $A^{\mathbb{K}}$ and that

$$\mathbf{M}_A^{\mathbb{K}} = (m_{\lambda,\mu})_{\lambda\in\Pi,\mu\in B_{\rhd}^{\mathbb{K}}} \qquad m_{\lambda\mu} = [V_\lambda^{\mathbb{K}} : L_\mu^{\mathbb{K}}]$$

is the modular decomposition matrix; this matrix is uni-triangular with respect to $\rhd$ by (i) and (iii).

Definition 2.2 ([HM10, Definition 2.1])**.** Suppose A^R is a $\mathbb{Z}$-graded R-algebra of finite rank over R. We say that A is a graded cellular algebra if the following conditions hold. The algebra is equipped with a datum $(\Pi, \mathcal{T}, C, \deg)$, where $(\Pi, \trianglerighteq)$ is the weight poset. For each $\lambda \in \Pi$ we have a finite set, denoted $\mathcal{T}(\lambda)$. There exist maps

$$C : \textstyle\coprod_{\lambda\in\Pi} \mathcal{T}(\lambda) \times \mathcal{T}(\lambda) \to A^R; \quad \text{and} \quad \deg : \textstyle\coprod_{\lambda\in\Pi} \mathcal{T}(\lambda) \to R$$

such that C is injective. We denote $C(\mathsf{S}, \mathsf{T}) = c_{\mathsf{ST}}^\lambda$ for $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$, and

1. Each c_{ST}^λ is homogeneous of degree $\deg(c_{\mathsf{ST}}^\lambda) = \deg(\mathsf{S}) + \deg(\mathsf{T})$, for $\lambda \in \Pi$ and $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$.
2. The set $\{c_{\mathsf{ST}}^\lambda \mid \mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda),\ \lambda \in \Pi\}$ is a R-basis of A^R.
3. If $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$, for some $\lambda \in \Pi$, and $a \in A^R$ then there exist scalars $r_{\mathsf{SU}}(a)$, which do not depend on T, such that
$$ac_{\mathsf{ST}}^\lambda = \sum_{\mathsf{U}\in\mathcal{T}(\lambda)} r_{\mathsf{SU}}(a) c_{\mathsf{UT}}^\lambda \pmod{A^{\rhd\lambda}},$$
where $A^{\rhd\lambda}$ is the R-submodule of A^R spanned by $\{c_{\mathsf{QR}}^\mu \mid \mu \rhd \lambda \text{ and } \mathsf{Q}, \mathsf{R} \in \mathcal{T}(\mu)\}$.
4. The $\Bbbk$-linear map $* : A^R \to A^R$ determined by $(c_{\mathsf{ST}}^\lambda)^* = c_{\mathsf{TS}}^\lambda$, for all $\lambda \in \Pi$ and all $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$, is an anti-isomorphism of A^R.

Given $\lambda \in \Pi$, the graded cell module $\Delta_A^R(\lambda)$ is the graded left A^R-module with basis $\{c_{\mathsf{S}}^\lambda \mid \mathsf{S} \in \mathcal{T}(\lambda)\}$. The action of A^R on $\Delta_A^R(\lambda)$ is given by

$$ac_{\mathsf{S}}^\lambda = \textstyle\sum_{\mathsf{U}\in\mathcal{T}(\lambda)} r_{\mathsf{SU}}(a) c_{\mathsf{U}}^\lambda,$$

where the scalars $r_{\mathsf{SU}}(a)$ are the scalars appearing in condition (3) of Definition 2.2. Suppose that $\lambda \in \Pi$. There is a bilinear form $\langle\ ,\ \rangle_\lambda$ on $\Delta_A^R(\lambda)$ which is determined by

$$c_{\mathsf{US}}^\lambda c_{\mathsf{TV}}^\lambda \equiv \langle c_{\mathsf{S}}^\lambda, c_{\mathsf{T}}^\lambda \rangle_\lambda c_{\mathsf{UV}}^\lambda \pmod{A^{\rhd\lambda}},$$

for any $\mathsf{S}, \mathsf{T}, \mathsf{U}, \mathsf{V} \in \mathcal{T}(\lambda)$. For every $\lambda \in \Pi$, we let $\langle\ ,\ \rangle_\lambda$ denote the bilinear form on $\Delta^R(\lambda)$ and let $\mathrm{rad}_A^R\langle\ ,\ \rangle_\lambda$ denote the radical of this bilinear form. We define $\Delta_A^{\mathbb{K}}(\lambda) = \Delta_A^R(\lambda) \otimes_R \mathbb{K}$ and extend all the notation above in the obvious manner. We set $\Lambda_{\rhd}^{\mathbb{K}} = \{\lambda \mid \mathrm{rad}^{\mathbb{K}}\langle\ ,\ \rangle_\lambda \neq \Delta_A^{\mathbb{K}}(\lambda)\}$ and we set $L_A^{\mathbb{K}}(\lambda) = \Delta_A^{\mathbb{K}}(\lambda)/\mathrm{rad}^{\mathbb{K}}\langle\ ,\ \rangle_\lambda$. By [HM10, Lemma 2.7], each module $L_A^{\mathbb{K}}(\lambda)$ is graded and simple, and in fact every irreducible module is of this form, up to grading shift. The passage between the (graded) cell and irreducible modules is recorded in the (graded) cellular decomposition matrix,

$$\mathbf{D}_A^{\mathbb{K}}(t) = (d_{\lambda\mu}(t))_{\lambda\in\Pi,\mu\in\Lambda_{\rhd}^{\mathbb{K}}} \qquad d_{\lambda\mu}(t) = \textstyle\sum_{k\in R} [\Delta_A^{\mathbb{K}}(\lambda) : L_A^{\mathbb{K}}(\mu)\langle k\rangle]\, t^k \in \mathbb{N}[t, t^{-1}].$$

This matrix is uni-triangular with respect to $\trianglerighteq$; thus if $A^{\mathbb{F}}$ is semisimple, we have that $\Lambda^{\mathbb{K}}_{\rhd}$ is a canonical basic set for $A^{\mathbb{K}}$. By Definition 2.1(i) canonical basic sets are unique and so, matching-up the labelling sets of semisimple modules and cell-modules, we immediately deduce the following:

Proposition 2.3. *Suppose that A^R is graded cellular with respect to $\rhd$. Suppose further that $A^{\mathbb{F}}$ is a semisimple $\mathbb{F}$-algebra and that $A^R \otimes \mathbb{K}$ has canonical basic set $B^{\mathbb{K}}_{\rhd}$. If $V^{\mathbb{F}}_\lambda \cong \Delta^{\mathbb{F}}_A(\lambda)$ for all $\lambda \in \Pi$, then $\Lambda^{\mathbb{K}}_{\rhd} = B^{\mathbb{K}}_{\rhd}$ and $\mathbf{D}^{\mathbb{K}}_A(t)|_{t=1} = \mathbf{M}^{\mathbb{K}}_A$.*

3. The quiver Hecke algebras

Let $\Bbbk$ be an arbitrary integral domain. We emphasise that the following presentation of the (quiver) Hecke algebra only depends on the reduction of the charge modulo e.

Definition 3.1 ([BK09a, KL09, Rou08]). Fix $e \in \mathbb{N}_{>1}$ and $\underline{s} \in (\mathbb{Z}/e\mathbb{Z})^\ell$. The quiver Hecke algebra, $\mathcal{H}^{\Bbbk}_n(\underline{s})$, is defined to be the unital, associative, finite-dimensional $\Bbbk$-algebra with generators

$$\{e(\imath) \mid \imath = (i_1, \dots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}, \tag{3.1}$$

subject to the relations

$$e(\imath)e(\jmath) = \delta_{\imath,\jmath} e(\imath); \tag{3.2}$$

$$\sum_{\imath \in (\mathbb{Z}/e\mathbb{Z})^n} e(\imath) = 1; \tag{3.3}$$

$$y_r e(\imath) = e(\imath) y_r; \tag{3.4}$$

$$\psi_r e(\imath) = e(s_{r,r+1}\imath)\psi_r; \tag{3.5}$$

$$y_r y_s = y_s y_r; \tag{3.6}$$

$$\psi_r y_s = y_s \psi_r \qquad \text{if } s \neq r, r+1; \tag{3.7}$$

$$\psi_r \psi_s = \psi_s \psi_r \qquad \text{if } |r-s| > 1; \tag{3.8}$$

$$y_r \psi_r e(\imath) = (\psi_r y_{r+1} - \delta_{i_r, i_{r+1}}) e(\imath); \tag{3.9}$$

$$y_{r+1} \psi_r e(\imath) = (\psi_r y_r + \delta_{i_r, i_{r+1}}) e(\imath); \tag{3.10}$$

$$\psi_r^2 e(\imath) = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ e(\imath) & \text{if } i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r) e(\imath) & \text{if } i_{r+1} = i_r - 1 \ \&\ e \neq 2, \\ (y_r - y_{r+1}) e(\imath) & \text{if } i_{r+1} = i_r + 1 \ \&\ e \neq 2, \\ (y_{r+1} - y_r)(y_r - y_{r+1}) e(\imath) & \text{if } i_{r+1} \neq i_r \ \&\ e = 2; \end{cases} \tag{3.11}$$

$$\psi_r \psi_{r+1} \psi_r = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1) e(\imath) & \text{if } i_r = i_{r+2} = i_{r+1} + 1 \ \&\ e \neq 2, \\ (\psi_{r+1}\psi_r\psi_{r+1} - 1) e(\imath) & \text{if } i_r = i_{r+2} = i_{r+1} - 1 \ \&\ e \neq 2, \\ (\psi_{r+1}\psi_r\psi_{r+1} + y_r - 2y_{r+1} + y_{r+2}) e(\imath) & \text{if } i_r = i_{r+2} \neq i_{r+1} \ \&\ e = 2, \\ (\psi_{r+1}\psi_r\psi_{r+1}) e(\imath) & \text{otherwise}; \end{cases} \tag{3.12}$$

for all admissible r, s, i, j. Finally, the cyclotomic relation: for $\imath \in (\mathbb{Z}/e\mathbb{Z})^n$, we have that

$$y_1^{\sharp\{s_m \mid s_m = i_1\}} e(\imath) = 0. \tag{3.13}$$

For ease of notation, we have excluded the $e = \infty$ case from the usual definition of the quiver Hecke algebra (note that the $e = \infty$ and $e > n$ algebras are isomorphic and so this is not important).

Theorem 3.2 ([BK09a, KL09, Rou08]). *We have a grading on $\mathcal{H}^{\Bbbk}_n(\underline{s})$ given by*

$$\deg(e(\imath)) = 0 \quad \deg(y_r) = 2 \quad \deg(\psi_r e(\imath)) = \begin{cases} -2 & \text{if } i_r = i_{r+1} \\ 1 & \text{if } i_r = i_{r+1} \pm 1 \\ 2 & \text{if } e{=}2 \text{ and } i_{r+1} \neq i_r \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.3 ([BK09a, Main Theorem]). *Let $\Bbbk$ be a field. The algebras $\mathcal{H}^{\Bbbk}_n(\underline{s})$ and $H^{\Bbbk}_n(\underline{s})$ are isomorphic as $\Bbbk$-algebras.*

For the remainder of the paper, we will work in the graded setting of $\mathcal{H}_n^{\Bbbk}(\underline{s})$. This is because we wish to prove Theorem A in the generality of an arbitrary integral domain $\Bbbk$ (for which the isomorphism of Theorem 3.3 fails).

4. Quiver Cherednik algebras

We now recall Webster's definition of the quiver (or diagrammatic) Cherednik algebra $\mathbb{A}_n^{\Bbbk}(\sigma)$ for $n \in \mathbb{N}$ and $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$. We shall see that the representation theoretic structure of $\mathbb{A}_n^{\Bbbk}(\sigma)$ is *heavily dependent* on the charge $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$. This is in stark contrast with the structure of the (quiver) Hecke algebra which we have seen is dependent only on the modulo e reduction: $(s_0, \dots, s_{\ell-1}) \in (\mathbb{Z}/e\mathbb{Z})^\ell$. In other words, the quiver Cherednik algebras have the extra combinatorial information of Section 1 baked into their definition. In Section 7, we shall apply the many "charged" Schur functors to these quiver Cherednik algebras in order to obtain many new presentations of the quiver Hecke algebra which encode the richer structures which cannot be detected using either the classical or KLR presentations. We have written this section in the style of a self-contained beginner's guide to the diagrammatic theory and have included many examples.

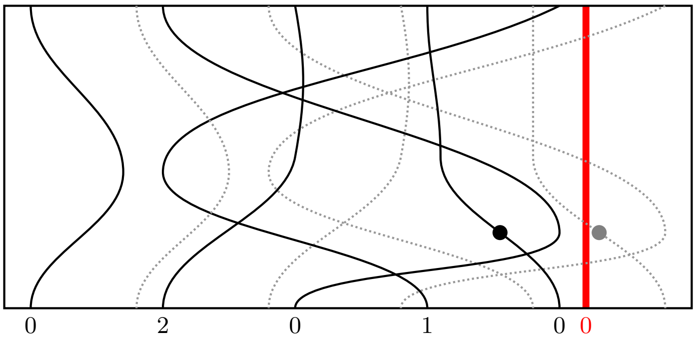


Figure 4. A σ-diagram, $A \in \mathbb{A}_5^{\Bbbk}(3;0)$, with northern and southern loading $\mathbf{I}_\omega^\sigma$ for $\omega = (1^5)$.

Definition 4.1. We define a σ-diagram of rank $n \in \mathbb{N}$ and type (μ, λ) to be a frame, $\mathbb{R} \times [0, 1]$, with n distinguished solid points on the northern and southern boundaries given by $\mathbf{I}_\mu^\sigma$ and $\mathbf{I}_\lambda^\sigma$ for $\lambda, \mu \in \mathscr{C}_n^\ell$ and a collection of solid strands each of which starts at a northern point and ends at a southern point. Each solid strand carries a residue, $i \in \mathbb{Z}/e\mathbb{Z}$, say (and we refer to this as a solid i-strand). We further require that each solid strand has a mapping diffeomorphically to $[0, 1]$ via the projection to the y-axis. Each solid strand can carry a finite number of dots. We draw

- (i) a "ghost i-strand" 1 unit to the right of each solid i-strand and a a "ghost dot" 1 unit to the right of each solid dot;
- (ii) vertical red lines with x-coordinate $\sigma_m - m/\ell \in \mathbb{Q}$ each of which carries a residue $s_m \in \mathbb{Z}/e\mathbb{Z}$ for $1 \leqslant m \leqslant \ell$ which we call a red s_m-strand.

Finally, we require that there are no triple points or tangencies involving any combination of strands, ghosts or red lines and no dots lie on crossings. We consider these diagrams equivalent if they are related by an isotopy that avoids these tangencies, double points and dots on crossings.

Definition 4.2. We define the degree of a σ-diagram to be the integer obtained by summing over the degrees of all the local neighbourhoods of the diagram, with each neighbourhood contributing to the degree as follows:

and their mirror images.

Definition 4.3. Let D be a σ-diagram. We define the northern (respectively southern) ordered residue sequence of D to be the element of $(\mathbb{Z}/e\mathbb{Z})^n$ given by reading the residues of the solid strands in D from left to right along the northern (respectively southern) edge of the frame.

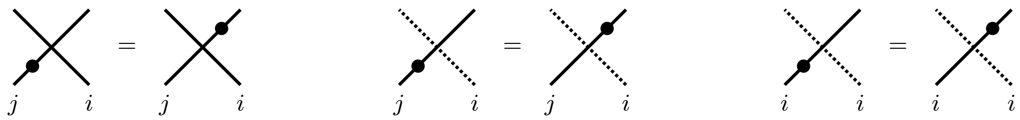
Definition 4.4. Let D be a σ-diagram. Suppose D has distinguished solid points on the northern and southern boundaries given by $\mathbf{I}^{\sigma}_{\mu}$ and $\mathbf{I}^{\sigma}_{\lambda}$ and northern and southern residue sequence given by $\imath$ and $\jmath \in (\mathbb{Z}/e\mathbb{Z})^n$ respectively. We say that a diagram D is reduced if (i) when read from south-to-north D traces out a bijection : $[\lambda] \to [\mu]$ using the *minimal* number of crossings between strands and (ii) D has no dots on any strands. We let ${}^{\imath}_{\mu}\mathcal{R}^{\jmath}_{\lambda}$ denote the set of all such reduced diagrams.

Example 4.5. Let $\sigma = (3; 0) \in \mathbb{N}_{>1} \times \mathbb{Z}$ and $\lambda = \mu = (1^5)$. In Figure 4 we picture a σ-diagram.

Definition 4.6. The quiver Cherednik algebra, $\mathbb{A}^{\Bbbk}_n(\sigma)$, is the associative $\Bbbk$-algebra generated (as a $\Bbbk$-module) by all inequivalent σ-diagrams modulo the local relations (A1) to (A13) below (here a local relation means one that can be specified by its effect on an arbitrarily small region of the diagram). The product a_1a_2 of two diagrams $a_1, a_2 \in \mathbb{A}^{\Bbbk}_n(\sigma)$ is then given by putting a_1 on top of a_2. This product is defined to be 0 unless the southern border of a_1 is given by the same loading as the northern border of a_2 with residues of strands matching in the obvious manner, in which case we obtain a new diagram with loading and labels inherited from those of a_1 and a_2.

Isotopy and dots through crossings. These are the easiest relations in the quiver Cherednik algebra. They also serve as a reminder that when we apply a relation in a region containing a solid/ghost strand, we must also also has an effect on its corresponding ghost/solid strand 1 unit to the right/left.

(A1) Any diagram may be deformed isotopically; that is, by a continuous deformation of the diagram which avoids tangencies, double points and dots on crossings.
(A2) Any solid dot can pass through a crossing of solid i- and j-strands for $i \neq j$ or an arbitrary crossing involving a ghost strand. Namely:



and their mirror images through reflection in the vertical axis hold.
(A3) We can pass a solid dot through a crossing of two like-labelled solid or ghost strands at the expense of an error term:

Ghost dots can pass through any crossing of strands (regardless of their residue) freely.

Example 4.7. For example in Figure 5 we apply relation (A3) locally to a region of the diagram containing the dot in Figure 4; however, moving the dot means we must also move the ghost dot (which must always be 1 unit to the right) and undoing the crossing of solid 0-strands means we must undo the corresponding crossing of ghost strands as in Figure 4.

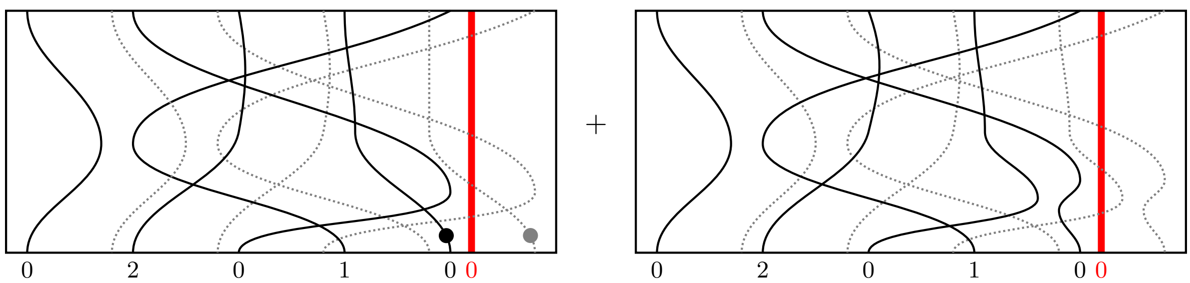
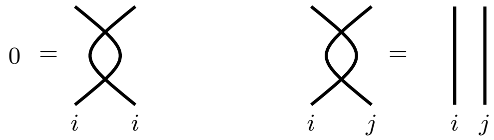


FIGURE 5. We apply relation (A3) to Figure 4 in order to move the dot through the crossing at the expense of an error term.

Undoing double-crossings. Now we consider how one can undo a pair of strands which cross and then cross again. The first of these relations, relation (A4), should be familiar from the classical KLR algebra.

(A4) For double-crossings of solid strands with $i \neq j$, we have the following local relations:



Performing relation (A4) implicitly involves undoing the corresponding double-crossing of ghost strands at the same time (which we do not picture) and vice versa.

Example 4.8. The leftmost diagram in Figure 5 has a double-crossing of two solid 0-strands and therefore this leftmost diagram is zero by relation (A4). (The observant reader might worry about the fact that a red 0-strand crosses the ghosts of these 0-strands — however, we shall see that this is not a problem in relation (A9).)

(A5) If $j \neq i-1$, then we can freely pass ghosts through solid strands. That is, we have the following local relations:

(A6) On the other hand, in the case where $j = i-1$, we have the following local relations:

Remark 4.9. It is worth noting that the local diagrammatic regions pictured in the left and right hand sides of relation (A6) do not have the same degree. This is because black dots carry degree 2 and ghost dots carry degree 0. However, we emphasise that by creating a ghost dot (in the local region pictured) we also create de facto solid dot (not pictured!) elsewhere in the diagram. Thus the overall degree of the diagrams is preserved (as one should expect!). An example of how this works in a wider diagram is pictured in Figure 6.

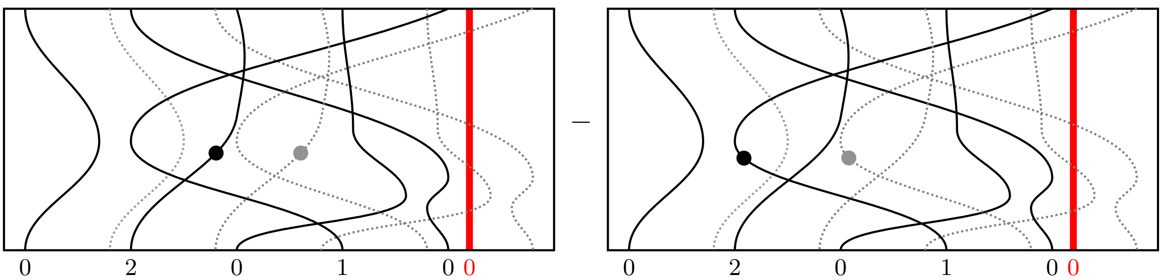
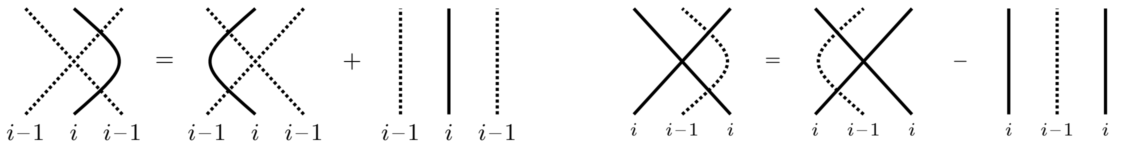


FIGURE 6. Undoing the double-crossing of the ghost 1-strand and a solid 2-strand in the rightmost diagram of Figure 5 using relation (A6)

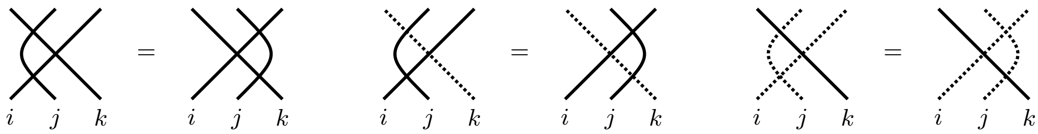
Example 4.10. The rightmost diagram in Figure 5 has a double-crossing of a ghost 1-strand and a solid 2-strand. We can continuously deform these strands until they are infinitesimally close together (without creating any tangencies of double points elsewhere in the diagram) and hence undo this double-crossing using relation (A6). This is depicted in Figure 6. (The other diagram in Figure 5 has this same double-crossing, but we have already seen that this diagram is zero using relation (A3).) We emphasise that there is a 0-ghost strand which passes through the region between the solid 1-strand and its ghost. It is clear that this does not hamper our ability to apply relation (A6) to the region containing the double-crossing of a ghost 1-strand and a solid 2-strand.

Pulling a strand through a crossing. We now consider the effect of pulling a strand through a pair of crossing strands. In other words, our graded versions of the classical braid relation.

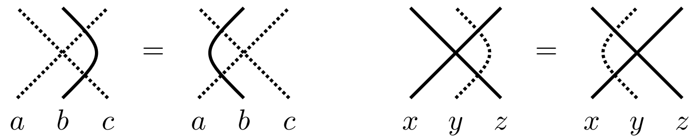
(A7) We can pull a solid i-strand through a $(i-1)$-ghost-crossing (or a ghost $(i-1)$-strand through a i-solid-crossing) at the expense of an error term.



(A8) All other triples of solid and ghost strands satisfy the naive braid relation. Diagrammatically, we have that



for any $i, j, k \in \mathbb{Z}/e\mathbb{Z}$ and their mirror images through reflection in the vertical axis hold. Performing the leftmost relation (A8) implicitly involves manipulating a braid of three ghost strands at the same time (which we do not picture) and vice versa. Furthermore,



and $a, b, c, x, y, z \in \mathbb{Z}/e\mathbb{Z}$ such that $\delta_{a,b-1,c} = 0$, $\delta_{x,y+1,z} = 0$.

Example 4.11. We now illustrate the effect of relations (A7) and (A8) by moving the 1-strand in the leftmost diagram in Figure 6 rightwards. We can do this (without incurring any error terms) until the solid 1-strand meets the crossing pair of ghost 0-strands. This is illustrated in Figure 7. We then apply relation (A7) to the diagram in Figure 7 to obtain the sum of diagrams in Figure 8.

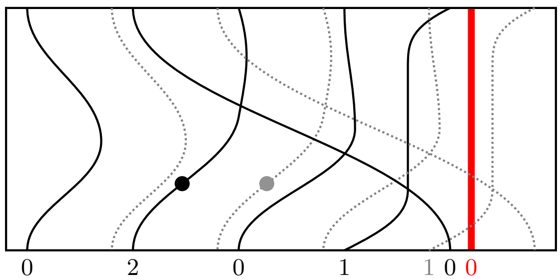


FIGURE 7. This diagram is obtained from the leftmost diagram in Figure 6 using non-interacting relations (isotopy, and moving the solid (respectively ghost) 1-strand rightwards without crossing any ghost 0-strand (respectively solid 2-strand) neither of which produces an error term.

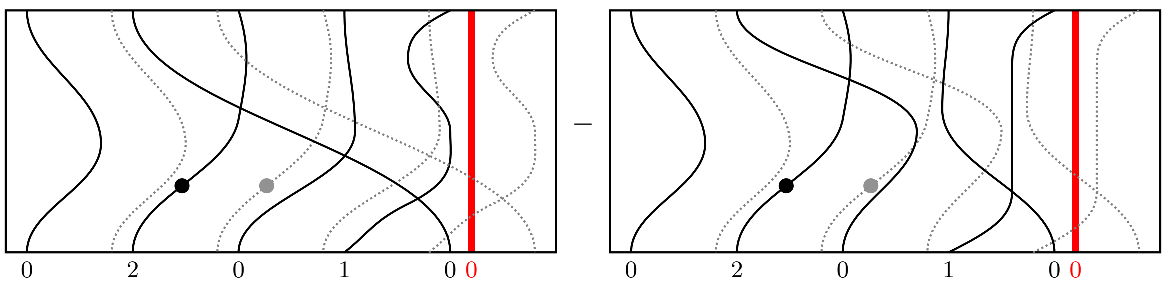
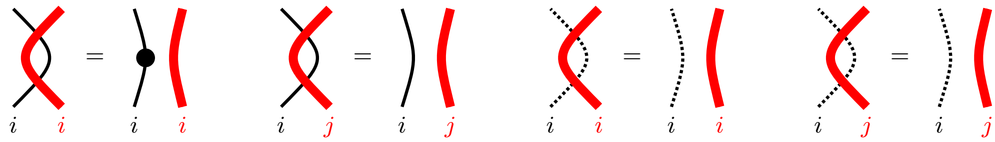


FIGURE 8. We apply relation (A7) to the diagram in Figure 7 thus passing the solid 1-strand through the crossing ghost 0-strands at the expensive of an error term.

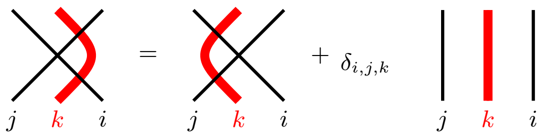
The red strands. We now consider the interactions between the red strands and the ghost and solid strands. One should think of these red strands and playing the analogue of the role of the cyclotomic relation for the classical KLR algebra.

(A9) Ghost strands and ghost dots may pass through red strands freely. For $i \neq j$, the solid i-strands may pass through red j-strands freely. If the red and solid strands have the same label, a dot is added to the solid strand when straightening. Diagrammatically, we have that

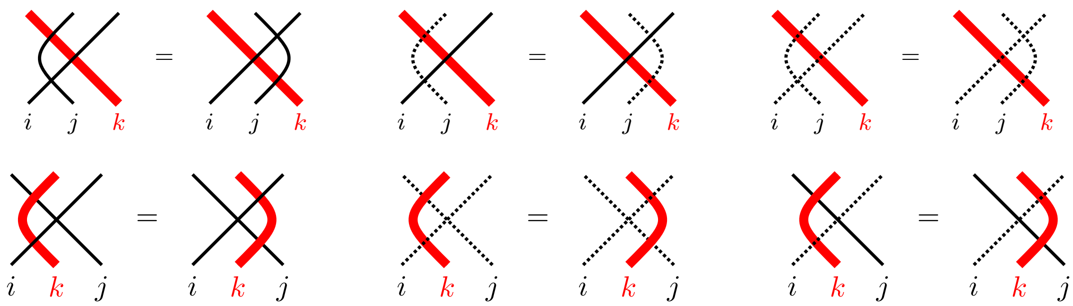


for $i \neq j$ and their mirror images through reflection through the vertical axis hold.

(A10) Solid crossings and dots can pass through red strands, with a correction term

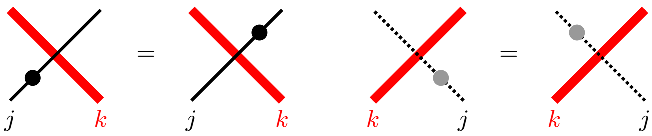


(A11) Any braid involving a red strand and not of the form in (A10) can be undone without cost. Diagrammatically, we have that



for any i, j, k and their mirror images through reflection in the vertical axis hold.

(A12) Finally, any solid or ghost dot can be pulled through a red strand without cost. Diagrammatically, we have that



for any j, k and their mirror images through reflection in the vertical axis hold. (We have not added the residues as they play no role here.)

The unsteady relation. Finally, we have the following non-local idempotent relation. Before doing so, we note that the unique (up to equivalency) element of ${}^{\imath}_{\mu}\mathcal{R}^{\imath}_{\mu}$ with *no crossing strands* is an idempotent by construction. We refer to any such diagram as the weight idempotent, $1^{\imath}_{\mu}$. When the northern (equivalently, southern) residue sequence of the weight idempotent is that of the box configuration, we simply write $1_{\mu} := 1^{\mathrm{res}(\mu)}_{\mu}$.

(A13) Any weight idempotent in which a solid strand is at least n units to the right of the rightmost red-strand is referred to as unsteady and set to be equal to zero.

Example 4.12. Consider the leftmost diagram in Figure 8. This diagram has a solid 1-strand which can be passed through the red 0-strand (without any error term) using relation (A9). This strand can then be pulled arbitrarily far to the right and hence is unsteady. Thus the leftmost diagram in Figure 8 is zero by relations (A9) and (A13).

Remark 4.13. We refer to relations (A1), (A2), (A5), (A8), (A11) and (A12), the latter relation in (A4), and the three rightmost relations in (A9) as non-interacting relations. These relations pull strands through one another in the naïve fashion (without acquiring error terms or dots).

5. The combinatorics of diagrams and box configurations

In this section we introduce the combinatorial language and corresponding diagrammatic relations needed for proving the main results of this paper.

5.1. **The Bruhat ordering.** We define a subword of $\underline{w} = s_{r_1,r_1+1}s_{r_2,r_2+1}\dots s_{r_\ell,r_\ell+1}$ to be a sequence $\mathbf{t} = (t_1, t_2, \dots, t_\ell) \in \{0,1\}^\ell$ and we set $\underline{w}^{\mathbf{t}} := s^{t_1}_{r_1,r_1+1}s^{t_2}_{r_2,r_2+1}\dots s^{t_\ell}_{r_\ell,r_\ell+1}$. We let $\leqslant$ denote the strong Bruhat order: namely $y \leqslant w$ if for some (or equivalently, every) reduced expression $\underline{w}$ there exists a subword t and a reduced expression $\underline{y}$ such that $\underline{w}^{\mathbf{t}} = \underline{y}$. We are now ready to define a new ordering on σ-diagrams; this ordering will be the key to our inductive proofs.

Definition 5.1. Let $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ and $\lambda, \mu \in \mathscr{C}^\ell_n$. Given $A \in 1^\imath_\lambda \mathbb{A}^\Bbbk_n(\sigma)1^\jmath_\mu$, we let $[A] \in \mathfrak{S}_{2n+\ell}$ denote the underlying word (in the Coxeter generators of $\mathfrak{S}_{2n+\ell}$) given by forgetting the types (solid, dashed, red) of strand and the distances between end points — in other words, simply viewing the diagram as a permutation in $\mathfrak{S}_{2n+\ell}$. Given two diagrams $A, A' \in 1^\imath_\lambda \mathbb{A}^\Bbbk_n(\sigma)1^\jmath_\mu$ we write $A' \trianglerighteq_\sigma A$ if $[A] \leqslant [A']$ in the Bruhat ordering on the permutations $[A], [A'] \in \mathfrak{S}_{2n+\ell}$. We let $\ell[A]$ denote the length of a reduced expression of $[A]$.

Remark 5.2. If $A, A' \in 1^\imath_\lambda \mathbb{A}^\Bbbk_n(\sigma)1^\jmath_\mu$ are equivalent diagrams, then the words $[A']$ and $[A]$ differ only by application of the commuting Coxeter relations ($s_{i,i+1}s_{j,j+1} = s_{j,j+1}s_{i,i+1}$ for $|i-j| > 1$). Therefore if two words differ by only the commuting Coxeter relations, we do not distinguish between these words.

For $n \in \mathbb{N}$ and $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ we emphasise that a σ-diagram from $\mathbb{A}^\Bbbk_n(\sigma)$ has n solid strands, n ghost strands and ℓ red strands; and our ordering considers all possible crossings of these $2n+\ell$ strands.

Example 5.3. The diagram from $\mathbb{A}^\Bbbk_5(0)$ in Figure 7 has 14 crossings between its 5 solid, 5 ghost, and 1 red strands. The underlying word is $s_{8,9}s_{4,5}s_{3,4}s_{5,6}s_{6,7}s_{4,5}s_{5,6}s_{7,8}s_{8,9}s_{7,8}s_{6,7}s_{7,8}s_{9,10}s_{10,11}s_{9,10}s_{8,9} \in \mathfrak{S}_{11}$. The righthand diagram of Figure 8 is obtained from the previous diagram by undoing a crossing of solid strands (and the corresponding crossing of their ghosts); this diagram has 12 crossings and the underlying permutation is $s_{8,9}s_{4,5}s_{3,4}s_{5,6}s_{6,7}s_{4,5}s_{6,7}s_{7,8}s_{9,10}s_{10,11}s_{9,10}s_{8,9} \in \mathfrak{S}_{11}$. Thus the latter diagram dominates the former in the Bruhat ordering.

5.2. **Brick combinatorics.** We require a language for discussing the effect of moving a single i-strand (and its ghost) through an idempotent 1_λ for $\lambda \in \mathscr{P}^\ell_n$. We can restrict our attention to understanding the boxes $(r,c,m) \in \lambda$ of residue $j \in \mathbb{Z}/e\mathbb{Z}$ such that $|j-i| \leqslant 1$. This leads us to introduce a combinatorial language of i-diagonals and bricks. This will be essential for the proofs of Theorems 6.8, 7.1 and 12.1, but can be skipped by the light-touch reader.

Definition 5.4. Let $\lambda \in \mathscr{C}^\ell_n$. Given $\kappa \in \mathbb{N}$ and $0 \leqslant m < \ell$, we refer to the set of boxes

$$\mathbf{D}_{m,\kappa} = \{(r,c,m) \in \lambda \mid \mathsf{ct}(r,c,m) \in \{\kappa-1, \kappa, \kappa+1\}\}$$

as the associated diagonal. If κ is greater than, less than, or equal to σ_m, we say that the diagonal is a right, left, or centred diagonal respectively. If $\lambda \in \mathscr{P}^\ell_n$, we say that a diagonal is addable, *removable*, or invisible if $\mathbf{D}$ contains an addable box of λ of content κ, a removable box of λ of content κ, or no such box respectively. Given $i \in \mathbb{Z}/e\mathbb{Z}$ we refer to any diagonal $\mathbf{D}_{m,\kappa}$ such that $i \equiv \kappa$ modulo e as an i-diagonal.

We shall now describe all ways of building i-diagonals of ℓ-partitions from the set of bricks $\mathbf{B}_k$ for $k = 1, \dots, 6$ depicted in Figure 10 and the empty brick, $\mathbf{B}_7$. We shall also require three distinct bricks $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ which represent the important box-configurations in which *some boxes are missing*. Namely, for a given i-box $(r,c,m) \in \lambda$ the cases $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ correspond to a missing box in λ to the south-west, south-east, or both respectively. These are depicted in Figure 10.

Fix $\lambda \in \mathscr{P}^\ell_n$ and consider some fixed component $1 \leqslant m \leqslant \ell$. We build an addable i-diagonal, $\mathbf{D}$, in this component by placing a a $\mathbf{B}_4$, $\mathbf{B}_5$, or $\mathbf{B}_7$ at the base (for diagonals to the right, left, or centred on the node $(1,1,m)$); we then place some number (possibly zero) of $\mathbf{B}_1$ bricks on top. If $\mathbf{D}$ is invisible then we place either a $\mathbf{B}_2$ or $\mathbf{B}_3$ brick on top of the addable i-diagonal. If $\mathbf{D}$ is removable then we place a $\mathbf{B}_6$-brick on top of the addable i-diagonal. Examples of how to construct such an i-diagonal are depicted in Figure 9.

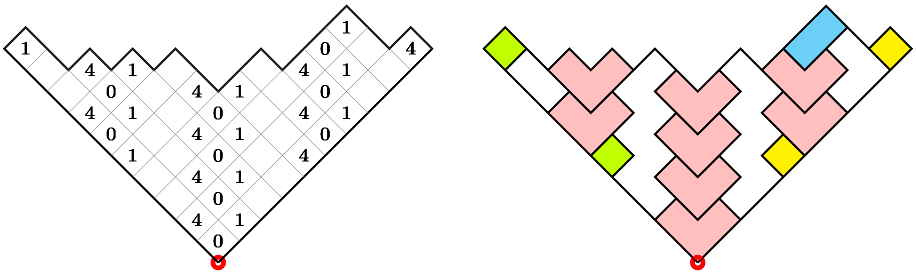


FIGURE 9. Examples of 0-diagonals for $e = 5$ (4 are addable and 1 is invisible). On the left we highlight all boxes in all 0-diagonals in the partition. On the right we illustrate how these diagonals are built up from bricks. Here $\mathbf{B}_1$ is pink, $\mathbf{B}_3$ is cyan, $\mathbf{B}_4$ is yellow, and $\mathbf{B}_5$ is green.

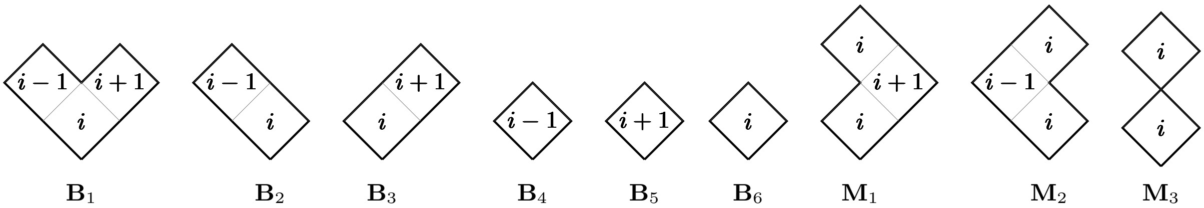


FIGURE 10. The bricks $\mathbf{B}_i$ and $\mathbf{M}_j$ for $1 \leqslant i \leqslant 6$ and $1 \leqslant j \leqslant 3$. The $\mathbf{B}_7$ brick is a single red i-strand (i.e., it corresponds to an empty box configuration).

5.3. **Brick diagrams.** We gather some easy results concerning the effect of pulling an i-strand through the diagram corresponding to one of these i-bricks. In order to go back and forth between σ-diagrams and box configurations, we make the following intuitive definition.

Definition 5.5. Associated to $\lambda \in \mathscr{C}_n^\ell$, $\imath \in (\mathbb{Z}/e\mathbb{Z})^n$, we have an idempotent $1_\lambda^\imath$ given by the diagram with northern/southern points $\mathbf{I}_\lambda^\sigma$, no crossing strands, and northern/southern residue sequence given by $\imath \in (\mathbb{Z}/e\mathbb{Z})^n$. Each brick, $\mathbf{B}_i$, is itself a box-configuration with associated idempotent $1_{\mathbf{B}_i}$ for $1 \leqslant i \leqslant 6$ (similarly for $\mathbf{M}_j$ for $1 \leqslant j \leqslant 3$). For $(r, c, m) \in \lambda$, we let $y_{(r,c,m)}1_\lambda^\imath$ be the diagram obtained by adding a dot to $1_\lambda^\imath$ on the strand labelled by the box (r, c, m).

Definition 5.6. For $\lambda \in \mathscr{C}_n^\ell$ and $\jmath \in (\mathbb{Z}/e\mathbb{Z})^n$, we let $\mathcal{Y}_\lambda^\jmath$ denote the subalgebra $\langle 1_\lambda^\jmath, y_{(r,c,m)}1_\lambda^\jmath \mid \text{for } (r, c, m) \in [\lambda]\rangle \subset \mathbb{A}_n^{\Bbbk}(\sigma)$.

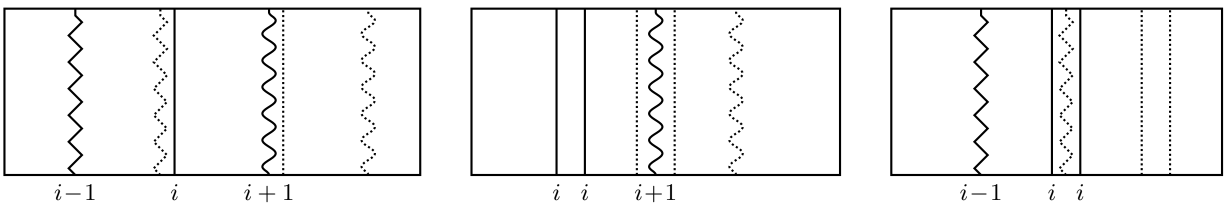


FIGURE 11. The diagrams $1_{\mathbf{B}_1}$, $1_{\mathbf{M}_1}$ and $1_{\mathbf{M}_2}$ respectively. We have applied isotopy to the strands to make it clearer which ghost belongs to which solid strand.

Definition 5.7. Given an i-diagonal, $\mathbf{D}$, we enumerate the bricks in $\mathbf{D}$ according to their height within the i-diagonal starting with the brick at the base of $\mathbf{D}$ first (which is one of $\mathbf{B}_4$, $\mathbf{B}_5$, or $\mathbf{B}_6$) and finishing with the top brick.

Remark 5.8. For ease of discussion, we assume $e > 2$ (the $e = 2$ case is similar, but the i-diagonals overlap and one must consider the diagonals in turn). Let S be a solid i-strand and suppose we wish to pull S and its ghost S' through an i-diagonal $\mathbf{D}$. One can factorise this calculation by considering each brick in the diagonal in turn. This is immediate from the definitions, but is slightly non-intuitive because we visualise the $(i-1)$-boxes as being to the left of the i-boxes which are in turn to the left of the $(i+1)$-boxes. However a momentary glance at Figure 12 reveals that this intuition is wrong: we will not encounter two successive i-strands at any point in the process, as they are separated by a ghost $(i-1)$-strand (and recall that S and S' both commute with the solid $(i-1)$-strand, it is *only* the *ghost* $(i-1)$-strand that is of interest!).

Indeed, the only strands of interest in $\mathbf{D}$ are the ghost $(i-1)$-strands, the solid i-strands, and the solid $(i+1)$-strands. For $\mathbf{D}$ as pictured in Figure 12, the ghost $(i-1)$-strands have x-coordinates $-3\varepsilon, -5\varepsilon$, the solid ghost i-strands have x-coordinates $-2\varepsilon, -4\varepsilon$ and the solid $(i+1)$-strands have x-coordinates $1-3\varepsilon, 1-5\varepsilon$. As we pull the strand S *and its ghost* through the strands labelled by boxes with x-coordinate $x - k\varepsilon$, the order in which these interactions occur is determined by $k \in \mathbb{N}$ and is independent of $x \in \mathbb{N}$. Thus one can factorise the calculation by considering each brick in the diagonal in turn, as claimed.

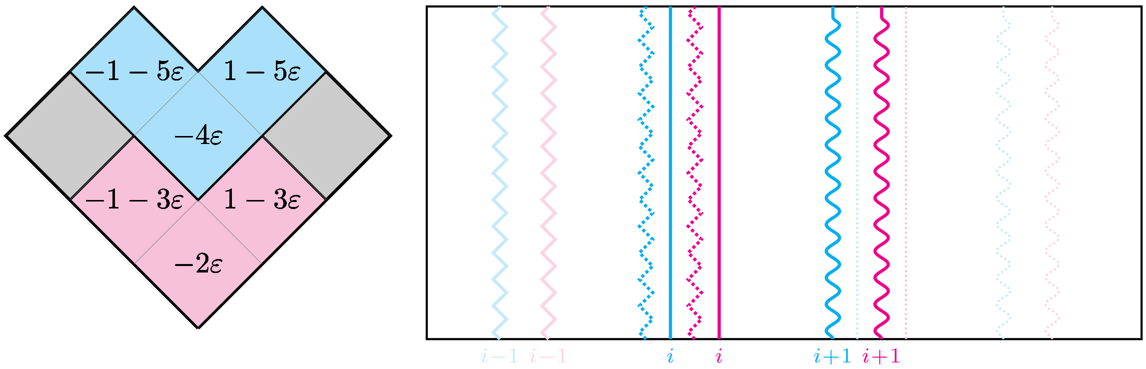
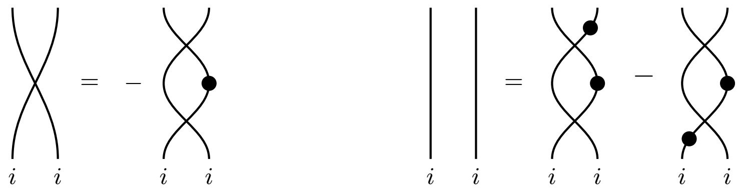


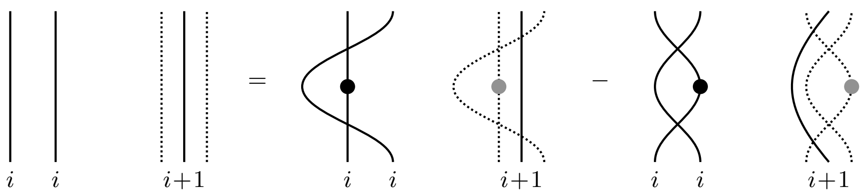
FIGURE 12. On the left we picture two $\mathbf{B}_1$ bricks (within the partition $(3^2,2)$) with the x-coordinates of their top nodes recorded in each box (we have assumed for ease of notation that the bottom of the lowest node has x-coordinate $x = 0$). On the right we picture the corresponding idempotent corresponding to this i-diagonal, emphasising (with bolder colour) the strands which do not commute with an i-strand S or its ghost S' (for $e > 2$). Notice that the i-strands are not adjacent, but rather they are separated by a ghost $(i-1)$-strand.

We will need to work by induction along the dominance ordering on box configurations. In order to do this, we need to be able to apply relations locally in a diagram and hence rewrite local regions of diagrams in terms of box configurations. The key to doing this is the following relations:



(5.1)

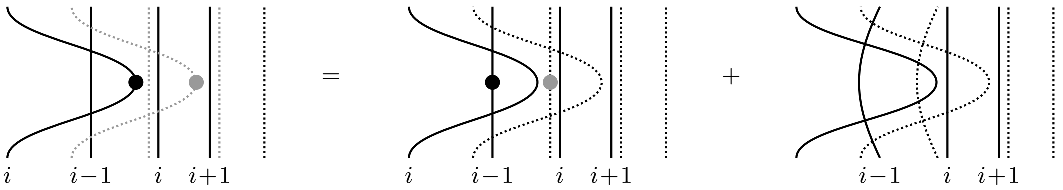
Both these relations follow by multiple applications of relation (A3). Let $\mathbf{M}_1$ be a brick containing the nodes $\{(r,c-1,m),(r+1,c-1,m),(r+1,c,m)\}$. Figure 13 illustrates how we can rewrite the diagram $1_{\mathbf{M}_1}$ by first applying the leftmost equality in relation (A7) followed by the leftmost equality in equation (5.1) (applied to both diagrams). We hence obtain the following:



(5.2)

Consider the idempotent corresponding to any 2×2 square of boxes $\{(r,c,m),(r-1,c,m),(r,c-1,m),(r-1,c-1,m)\}$ and place a dot on the strand labelled by (r,c,m); the diagram and the 2×2-array is depicted on the lefthand-side of the equation in Figure 14. We can pull this dotted

i-strand and its ghost rightwards through the ghost $(i-1)$-strand using relation (A6) to obtain a dotted and an undotted diagram. We hence obtain the following:



(5.3)

In order to associate equation (5.2) and (5.3) to manipulations of brick diagrams, we must consider the intersection of the line $y = 1/2$ with these diagrams. These are idempotents corresponding to certain bricks; we depict the corresponding bricks in Figures 13 and 14.

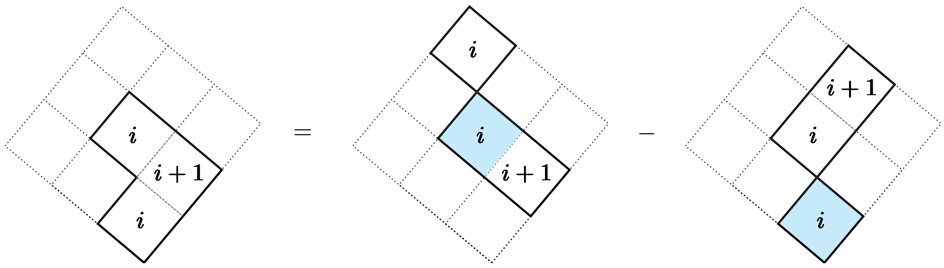


Figure 13. The brick diagrams depict the intersection of the σ-diagrams in equation (5.2) with the line $y = 1/2$. We shade the box corresponding to the decorated strand in each case.

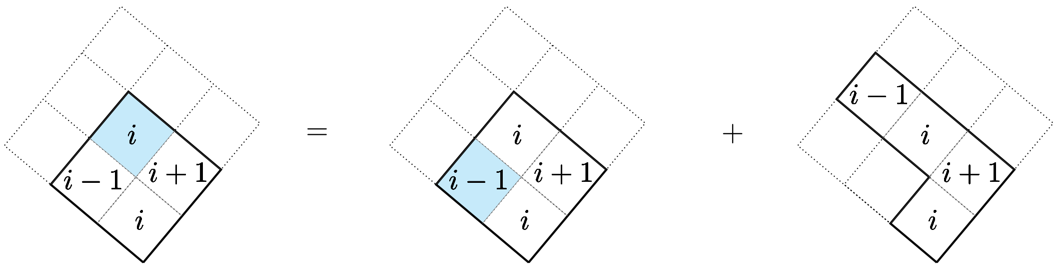


Figure 14. The brick diagrams depict the intersection of the σ-diagrams in equation (5.3) with the line $y = 1/2$. We shade the box corresponding to the decorated strand in each case.

The following technical definition will allow us to handle the inductive error terms of equation (5.2) and (5.3) by induction on the dominance ordering. In particular, notice that if we set the node (r, c, m) to be the missing node in the $\mathbf{M}_1$ brick in equation (5.1) then the maps $\phi^{X_1}_{r,c,m}$ and $\phi^{X_2}_{r,c,m}$ describe the intersection of the line $y = 1/2$ with the two terms on the righthand-side of equation (5.2). Similarly $\phi^{\mathbf{B}_1}_{(r,c,m)}$ describes the second term on the righthand-side of Figure 14.

Definition 5.9. Let $1 \leqslant r, c \leqslant n$ and $0 \leqslant m < \ell$. Given $\lambda \in \mathscr{C}^\ell_n$, we set $\phi(\lambda) \in \mathscr{C}^\ell_n$ to be the box configuration $\phi(\lambda) = \{\phi(r', c', m') \mid (r', c', m') \in \lambda\}$ for ϕ any one of the three following maps:

$$\phi^{N}_{(r,c,m)}(r', c', m') = \begin{cases} (r'+1, c'+1, m') & \text{if } m = m' \text{ and } r' > r \text{ and } c' > c \\ (r'+1, c'+1, m') & \text{if } (r'+1, c'+1, m') = (r, c, m) \\ (r', c', m') & \text{otherwise.} \end{cases}$$

$$\phi^{NE}_{(r,c,m)}(r', c', m') = \begin{cases} (r'+1, c'+1, m') & \text{if } m = m' \text{ and } r' \geqslant r \text{ and } c' > c \\ (r'+1, c'+1, m') & \text{if } (r'+1, c'+1, m') = (r, c, m) \\ (r', c', m') & \text{otherwise.} \end{cases}$$

$$\phi^{NW}_{(r,c,m)}(r', c', m') = \begin{cases} (r'+1, c'+1, m') & \text{if } m = m' \text{ and } r' > r \text{ and } c' \geqslant c \\ (r'+1, c'+1, m') & \text{if } (r'+1, c'+1, m') = (r, c, m) \\ (r', c', m') & \text{otherwise.} \end{cases}$$

6. The integral cellular basis of the quiver Cherednik algebra

In this section, we prove that $\mathbb{A}_n^{\Bbbk}(\sigma)$ is cellular for any $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ and over $\Bbbk$ an arbitrary integral domain, we define the associated Schur functor E_ω^σ relating the quiver Cherednik and Hecke algebras, and we generalise and strengthen the structural results of [BKW11, Web17] and [KL09, Section 2.3]. The framework we developed in Subsections 5.2 and 5.3 allows us to proceed by induction on $(\mathscr{C}_n^\ell, \rhd_\sigma)$. In Theorem 6.17 we directly match-up the presentations of the KLR algebra and (a subalgebra of) the quiver Cherednik algebra for the first time, this should be of independent interest. Over $\mathbb{C}$, cellularity of $\mathbb{A}_n^{\mathbb{C}}(\sigma)$ is proven in [Web17] by applying the isomorphism in [Web17, Theorem 4.5] to the ungraded versions of these algebras (this isomorphism generalises that of [BK09a] and only holds for $\mathbb{C}$). We take this opportunity to add a little flesh to the bones of the ideas [Web17]. We also prove a number of new structural results concerning the action of the algebra $\mathbb{A}_n^{\Bbbk}(\sigma)$ on the cellular basis (generalising [BKW11]).

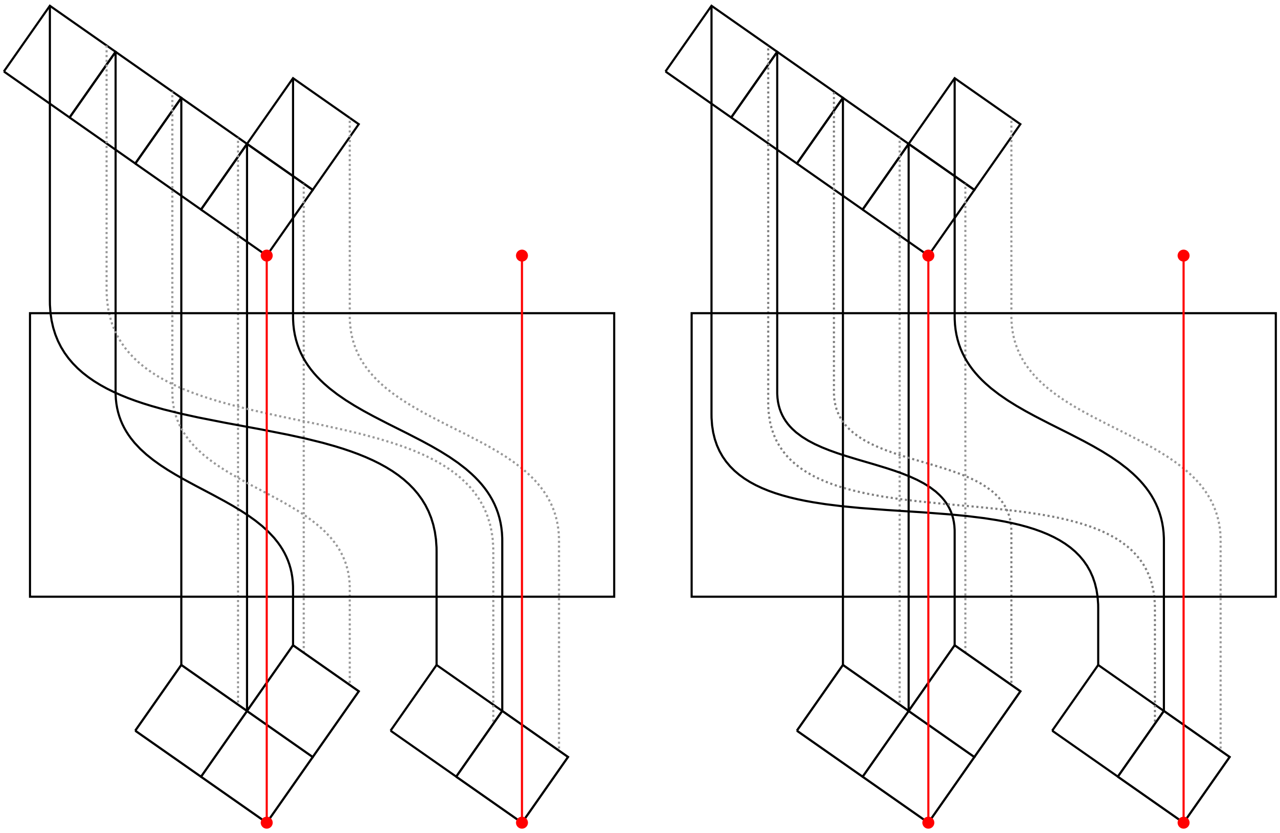


Figure 15. Two distinct diagrams A_{S} and A'_{S} associated to the tableau $\mathsf{S} \in \mathrm{SStd}_{(3;4,0)}(((1^2),(2,1)),(\varnothing,(2,1^3)))$ as in Figure 3.

Definition 6.1. Given S a tableau of shape λ and weight μ, we let $A_{\mathsf{S}} \in 1_\mu^\imath \mathbb{A}_n^{\Bbbk}(\sigma) 1_\lambda^{\mathsf{res}(\lambda)}$ denote any reduced diagram tracing out the bijection $\mathsf{S} : [\lambda] \to [\mu]$. Given S, T a pair of tableaux of shape λ (and possibly distinct weights) we set $A_{\mathsf{ST}} = A_{\mathsf{S}} A_{\mathsf{T}}^*$ where A_{T}^* is the diagram obtained from A_{T} by flipping it through the horizontal axis.

6.1. **Right justification.** The following total order refines the dominance order from Section 1. Given $i \in \mathbb{Z}/e\mathbb{Z}$ and (r,c,m) and (r',c',m') two i-boxes, we write $(r,c,m) \preceq (r',c',m')$ if

(i) $\mathsf{ct}(r,c,m) < \mathsf{ct}(r',c',m')$ or
(ii) $\mathsf{ct}(r,c,m) = \mathsf{ct}(r',c',m')$ and either $m > m'$ or $m = m'$ and $r + c \leqslant r' + c'$.

For $\lambda, \mu \in \mathscr{C}_n^\ell$, we write $\mu \preceq_\sigma \lambda$ if there is a bijective map $\mathsf{A} : [\lambda] \to [\mu]$ such that $\mathsf{A}(r,c,m) \preceq_\sigma (r,c,m)$ for all $(r,c,m) \in \lambda$. Given $\lambda, \mu \in \mathscr{C}_n^\ell$ we note that $\lambda \lhd_\sigma \mu$ implies $\lambda \prec_\sigma \mu$. We write $(r,c,m) \prec_{\mathrm{co}} (r',c',m')$ if $(r,c,m) \prec_{\mathrm{co}} (r',c',m')$ and there does not exist any (r'',c'',m'') such that $(r,c,m) \prec (r'',c'',m'') \prec (r',c',m')$. In which case, we say that (r,c,m) and (r',c',m') are consecutive and say that the latter immediately follows the former.

Remark 6.2. Recall the definition of ϕ from Definition 5.9. We have that $\lambda \succ_\sigma \phi(\lambda)$, however λ and $\phi(\lambda)$ are not relatable in the dominance ordering. In particular $\mu \rhd_\sigma \lambda$ if and only if $\mu \rhd_\sigma \phi(\lambda)$.

Definition 6.3. Given $\xi \in \mathscr{P}_n^\ell$ and $(r,c,m) \in \xi$ we say that (r,c,m) is right-justified if one of the following holds: (i) $(r-1,c,m) \in \xi$ (ii) $(r,c-1,m) \in \xi$ (iii) $(r-1,c-1,m) \in \xi$ (iv) $r = c = 1$. We say that $\xi \in \mathscr{C}_n^\ell$ is right-justified if and only if every $(r,c,m) \in \xi$ is right justified.

Let $\xi \in \mathscr{C}_n^\ell$ and suppose that $(r,c,m) \in \xi$ is not right justified. We set (r',c',m') equal to the box immediately following (r,c,m) in the order $\prec_\sigma$ and $\xi' = (\xi \cup \{(r',c',m')\}) \setminus \{(r,c,m)\}$. We say that ξ' is obtained from ξ by right-justifying the box (r,c,m). More generally, suppose there exists a chain

$$\xi = \xi^{(0)} \prec_\sigma \xi^{(1)} \prec_\sigma \cdots \prec_\sigma \xi^{(r)} = \xi'$$

and suppose that $\xi^{(i+1)}$ is obtained from $\xi^{(i)}$ by right-justifying some box $(r_i,c_i,m_i) \in \xi^{(i)}$; then we say that ξ' is is obtained from ξ by right-justification.

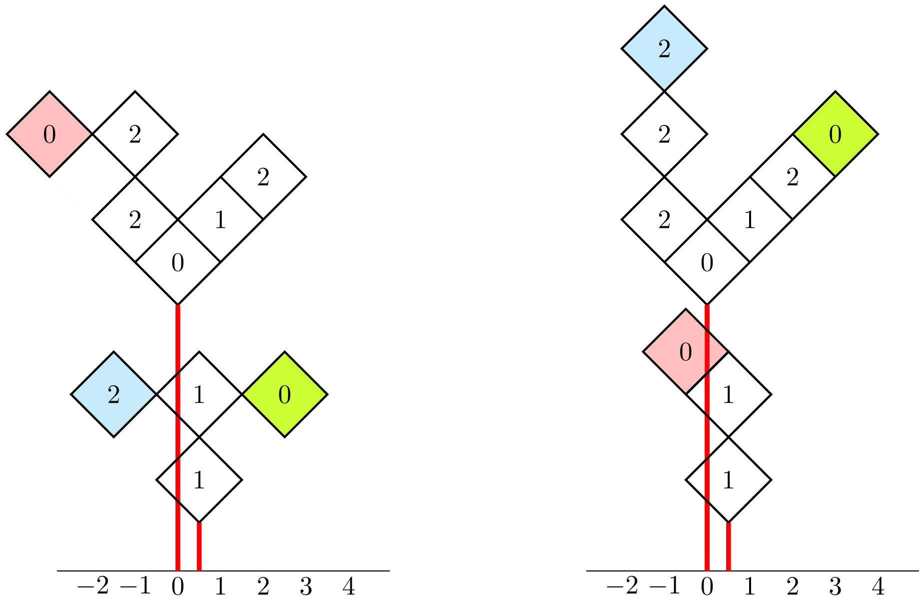


FIGURE 16. For $e = 3$ and $\sigma = (0,1) \in \mathbb{Z}^2$ we depict a box configuration and its right justification, respectively.

6.2. **A spanning set of the algebra.** In this subsection we provide a spanning set for $\mathbb{A}_n^{\Bbbk}(\sigma)$ and provide analogues of a number of results from [BKW11]. This section is inspired by the ideas of [Web17, BKW11]. In this section we shall assume, without loss of generality, that $s_1 \geqslant s_2 \geqslant \ldots \geqslant s_m$ (this is simply for ease of notation when describing the maximal and minimal elements of the dominance order and one can simply reorder the charge if necessary).

Proposition 6.4. *Any σ-diagram $A \in 1_\lambda^{\imath}\mathbb{A}_n^{\Bbbk}(\sigma)1_\mu^{\jmath}$ can be written in the form $A = a1_\xi a'$ for ξ a right-justified box configuration such that $\xi \succeq_\sigma \lambda,\mu$. Moreover any idempotent 1_ξ for $\xi \in \mathscr{C}_n^\ell$ belongs to $\mathbb{A}_n^{\Bbbk}(\sigma)1_{\xi'}\mathbb{A}_n^{\Bbbk}(\sigma)$ for $\xi' \succeq \xi$ obtained from ξ by right justification.*

Proof. Let $y \in [2\varepsilon, 1-2\varepsilon]$. Let S denote a solid or red strand in the diagram and let $x(S)$ denote the x-coordinate of this strand at the point where it intersects the line $\{y\} \times \mathbb{R}$. We write $S_1 \lessdot S_2$ if

(i) S_1 and S_2 are solid-strands of the same residue and $0 < x(S_1) - x(S_2) \leqslant 2\varepsilon$;
(ii) S_1 is a solid $(i+1)$-strand S_2 is a solid i-strand and $1 < x(S_1) - x(S_2) \leqslant 1 + \varepsilon$;
(iii) S_1 is a solid $(i-1)$-strand S_2 is a solid i-strand and $-1 \leqslant x(S_1) - x(S_2) < \varepsilon - 1$;
(iv) S_1 is a red i-strand S_2 is a solid i-strand and $0 < x(S_1) - x(S_2) \leqslant 2\varepsilon$.

We extend this to a partial ordering on strands by taking the transitive closure. We first explain how, applying *only the non-interacting relations* to our diagram A, we can group the strands into $\lessdot$-equivalence classes. We then show that these equivalence classes correspond to the components of a box-configuration. The reader might already see how the definition of right justification of boxes intuitively captures this process (such a reader is invited to skip the rest of this proof, as it is merely an in depth description of this process). We remark that cases (i) to (iii) correspond to i-bricks $\mathbf{M}_3$, $\mathbf{B}_3$, $\mathbf{B}_2$.

Consider the region $A \cap (\mathbb{R} \times [y-2\varepsilon, y+2\varepsilon])$ of our diagram A. We may assume that there are no crossing strands in this region and moreover that all strands in this region are vertical lines (by applying local isotopy (A1) if necessary, we can move any crossings above or below the region and straighten all the strands within the region).

Let S_2 denote a strand in the diagram $A \cap (\mathbb{R} \times [y-2\varepsilon, y+2\varepsilon])$. We pull the strand S_2 rightwards under the process outlined above. Let S_1 denote any strand in $A \cap (\mathbb{R} \times [y-2\varepsilon, y+2\varepsilon])$ which S_2 interacts with during this process of being pulled rightwards. Then either (i) the S_2-strand passes through the S_1-strand using the non-interacting relations or (ii) the S_2-strand comes to a halt at a point such that the strands S_1 and S_2 are in one of cases (i) to (iv) above. Having obtained A' (which we assume is not zero under relation (A13)) by pulling all solid strands as far right as possible in this manner (while keeping the red strands fixed) we find that the solid and red strands in the diagram A' have naturally gathered into ℓ distinct $\lessdot$-connected components (each containing precisely 1 red strand). This is simply because (i) the difference between the x-coordinates of two red strands is equal to an element of $\frac{1}{\ell}\mathbb{Z}$ (ii) $\varepsilon \ll \frac{1}{2n\ell}$ (iii) if $S_1 \lessdot S_2 \lessdot \ldots \lessdot S_k$ for $1 \leqslant k \leqslant n$, then $|x(S_k) - x(S_1)| \leqslant 2n\varepsilon \pmod{\mathbb{Z}}$ using (i) to (iv).

We now consider the mth $\lessdot$-connected component, denoted Θ_m, of strands containing the vertical red-strand with x-coordinate $\sigma_m - m/\ell$ and residue $s_m \in (\mathbb{Z}/e\mathbb{Z})$. If Θ_m contains no vertical strands, then this $\lessdot$-connected component corresponds to an empty component of the box configuration and we are done. Assume Θ_m contains at least one solid strand (i.e. $|\Theta_m| > 1$). Then by (i) to (iv) there exists at least one solid s_m-strand, $S_1 \in \Theta_m$, in the region $[\sigma_m - m/\ell - 2\varepsilon, \sigma_m - m/\ell)$. If $|\Theta_m| > 2$, then by (i) to (iv) there exists $S_2 \in \Theta_m$ such that one of the following holds (i) S_2 is a solid s_m-strand with $x(S_2) \in [\sigma_m - m/\ell - 4\varepsilon, \sigma_m - m/\ell)$ or (ii) S_2 is a solid (s_m-1)-strand with $x(S_2) \in [\sigma_m - m/\ell - 1 - 3\varepsilon, \sigma_m - m/\ell - 1)$ or (iii) S_2 is a solid (s_m+1)-strand with $x(S_2) \in [\sigma_m - m/\ell - 1 - 3\varepsilon, \sigma_m - m/\ell - 1)$. Continuing in this fashion, we find that for $x \in \mathbb{Z}/e\mathbb{Z}$ the solid strands in the region $\cup_{k \in \mathbb{N}} [ke + x - m/\ell - 2n\varepsilon, ke + x - m/\ell)$ are precisely the solid x-strands in Θ_m. By isotopy, we can assume that each strand in Θ_m has maximal x-coordinate such that the strands are still related under $\gtrdot$. In so doing, we find that each i-strand intersects the line y at some point equal to $\mathbf{I}^{\sigma}_{(r,c,m)}$ for some i-box (r, c, m). We now restrict to the region $X = A' \cap (\mathbb{R} \times [y-\varepsilon, y+\varepsilon])$ and by isotopy we can assume that all strands in this region are vertical. We hence have that $X = \mathbf{1}_\xi$ for some $\xi \in \mathscr{C}^\ell_n$ and that any pair of vertical strands in $X = \mathbf{1}_\xi$ is of the form (i), (ii), (iii), or (iv) above. In particular if $(r, c, m) \in \xi$ then this implies that at least one of $(r-1, c, m), (r, c-1, m), (r-1, c-1, m) \in \xi$ or $r = c = 1$ as required.

For the second claim, let $X = \mathbf{1}_\xi$ for $\xi \in \mathscr{C}^\ell_n$. Moving a strand corresponding to $(r, c, m) \in \xi$ rightwards using non-interacting relations (in the process above!) corresponds to the process of right-justifying the box $(r, c, m) \in \xi$. The result follows. □

Corollary 6.5. *The algebra $\mathbb{A}^{\Bbbk}_n(\sigma)$ is finitely generated by the set of all reduced diagrams and the elements $y_{(r,c,m)}\mathbf{1}_\mu$ for $\mu \in \mathscr{C}^\ell_n$.*

Proof. As in the proof of Proposition 6.4, we can make successive horizontal cuts to any $A \in \mathbb{A}^{\Bbbk}_n(\sigma)$ until we have rewritten $A = A_1 A_2 \ldots A_k$ such that each A_i is either reduced (with northern and southern loadings given by box configurations) or is obtained by adding a single dot to a strand in a weight idempotent. □

Remark 6.6. The upshot of Proposition 6.4 is that we can organise our proofs (which will involve manipulating σ-diagrams) by induction on the dominance order on box-configurations. In future proofs, we can gloss over any steps involving the non-interacting relations and instead focus on manipulating σ-diagrams corresponding to right-justified box-configurations.

Proposition 6.7. *For $\mu \in \mathscr{C}^\ell_n$ with $(r,c,m) \in \mu$ and $\lambda \in \mathscr{C}^\ell_n \setminus \mathscr{P}^\ell_n$, we have that*

$$y_{(r,c,m)}1_\mu \in \mathbb{A}_n^{\rhd\mu}(\sigma) \qquad 1_\lambda \in \mathbb{A}_n^{\rhd\lambda}(\sigma)$$

where $\mathbb{A}_n^{\rhd\mu}(\sigma) = \mathbb{A}_n^{\Bbbk}(\sigma)\langle 1_\nu \mid \nu \in \mathscr{P}^\ell_n, \nu \rhd \mu\rangle \mathbb{A}_n^{\Bbbk}(\sigma)$ and the elements are defined in Definition 5.5.

Proof. We shall prove both statements simultaneously by (intertwined) reverse induction on the dominance ordering on box-configurations. Let $\xi := ((n), \varnothing, \dots, \varnothing) \in \mathscr{C}^\ell_n$. If $\xi' \in \mathscr{C}^\ell_n$ is such that $\xi' \rhd \xi$, then it easy to see that $1_{\xi'} = 0$ under relation (A13). Applying relation (A6) or (A9) to $y_{(r,c,m)}1_\xi$ as necessary, we have that $y_{(r,c,m)}1_\xi = 0$ by relation (A13). Therefore the base case for induction holds. Now, assume that $\xi \in \mathscr{C}^\ell_n$ is right-justified (by Proposition 6.4) and suppose that the result has been proven for all box-configurations strictly more dominant than ξ. We refine our induction by the natural ordering on $r+c \geqslant 2$. We assume that for all $r'+c' < r+c$, we have

(α) if $(r',c',m') \notin \xi$ this implies $(r'+1,c',m) \notin \xi$ and $(r',c'+1,m) \notin \xi$.
(β) $y_{(r',c',m')}1_\xi \in \mathbb{A}_n^{\rhd\xi}(\sigma)$.

We first check the base case for our inductive assumptions for which $r+c=2$.

(a) We have that $r+c=2$ and $(1,1,m) \notin \xi$ for some $0 \leqslant m < \ell$. We can pull the strand labelled by $(1,2,m)$ or $(2,1,m)$ to the right using the non-interacting relations until the strand labelled by $(1,2,m)$ or $(2,1,m)$ encounters the $(i+1)$- respectively $(i-1)$-box immediately following $(1,2,m)$ or $(2,1,m)$. The resulting diagram factors through an idempotent $1_{\xi'}$ for $\xi' \in \mathscr{C}^\ell_n$ such that $\xi' \rhd \xi$ and the result follows by induction on the dominance ordering on box-configurations.
(b) We have that $r+c=2$, $(1,1,m) \in \xi$, and we have that a dot on the strand labelled by $(1,1,m) \in \xi$ for some $0 \leqslant m < \ell$. We apply relation (A9) to the solid strand labelled by $(1,1,m)$ and the red strand with x-coordinate $\sigma_m - m/\ell$. We can now pull the solid strand rightwards to obtain a more dominant box configuration; the result follows by induction on the $\rhd_\sigma$ ordering on $\mathscr{C}^\ell_n$.

Now for the inductive step.

(A) We have that $(r,c,m) \notin \xi$ for some $r+c > 2$, but that $(a,b,m) \in \xi$ for all $2 \leqslant a+b < r+c$. If $(1,c+1,m) \in \xi$ and $(1,c,m) \notin \xi$ (similarly if $(r+1,1,m) \in \xi$ and $(r,1,m) \notin \xi$) then then one can argue as in the $r=c=1$ case above. We now assume this is not the case. Since ξ is right justified, there are three cases to consider (corresponding to the $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ bricks).
  (**M**1) Suppose $(r-1,c+1,m), (r,c+1,m) \in \xi$ (and note that $(r-1,c,m) \in \xi$ by induction). We apply the relation depicted in equation (5.2) and Figure 13 to the the triple of strands in 1_ξ labelled by $(r,c+1,m)$, $(r-1,c+1,m)$ and $(r-1,c,m)$. We hence obtain a sum of two diagrams $X_2 - X_1$. We have that $X_1 \cap (\mathbb{R}\times\{1/2\}) = y_{(r-1,c,m)}1_{\xi'}$ for $\xi' = \phi^{NE}_{(r-1,c+1,m)}(\xi)$. We have that $X_2 \cap (\mathbb{R}\times\{1/2\}) = 1_{\xi''}$ where $\xi'' = \phi^{N}_{r-1,c,m}(\xi)$ and we note that $(r-1,c,m) \notin \xi''$. Therefore X_1 and X_2 factor through idempotents strictly more dominant than ξ' and ξ'' respectively (by induction on $r+c$) and therefore both factor through an idempotent strictly more dominant than ξ by Remark 6.2.
  (**M**2) Suppose $(r+1,c,m) \in \xi$ and and $(r+1,c-1,m) \in \xi$ (and $(r,c-1,m) \in \xi$ by induction). This case is similar to (**M**1) except that we use the *rightmost* (not the leftmost) relation (A6) in the analogue of equation (5.2).
  (**M**3) Now suppose that $(r-1,c+1,m) \notin \xi$ (the case $(r+1,c-1,m) \notin \xi$ is identical). We apply the rightmost equation of equation (5.1) to the strands labelled by $(r-1,c,m)$ and $(r,c+1,m)$ to obtain a sum of two diagrams $Y_1' - Y_2'$ which both factor through the diagram $y_{(r-1,c,m)}1_\xi$. Therefore both factor through an idempotent strictly more dominant than ξ by induction on $(r+c)$.
(B) If $r=1$ or $c=1$ then one can argue as in the $r=c=1$ case above with the exception that we replace the reference to relation (A9) with (A6). For $r,c>1$, our inductive assumption implies $(r-1,c,m), (r,c-1,m), (r-1,c-1,m) \in \xi$. We apply the relation depicted in equation (5.2) to the the quadruple of strands in 1_ξ labelled by $(r-1,c,m), (r,c-1,m)$, $(r-1,c-1,m)$, and (r,c,m). We hence obtain a sum of two diagrams Z_1+Z_2. We have that $Z_1 \cap (\mathbb{R}\times\{1/2\}) = y_{(r,c-1,m)}1_\xi$. We have that $Z_2 \cap (\mathbb{R}\times\{1/2\}) = 1_{\xi'}$ where $\xi' = \phi^{NW}_{r,c-1,m}(\xi)$ and we note that $(r,c-1,m) \notin \xi'$. Therefore Z_1 and Z_2 factor through idempotents strictly more dominant than

ξ and ξ' respectively (by induction on $r+c$) and so both factor through an idempotent strictly more dominant than ξ (by Remark 6.2). $\square$

Theorem 6.8. *We let $\lambda^{(0)} \geqslant \lambda^{(1)} \geqslant \lambda^{(2)} \geqslant \ldots \geqslant \lambda^{(m)}$ denote any total refinement of the order, $\rhd_\sigma$, on $\mathscr{P}^\ell_n$. The $\Bbbk$-algebra $\mathbb{A}^{\Bbbk}_n(\sigma)$ has a filtration*

$$0 \subset \mathbb{A}_n^{\geqslant\lambda^{(0)}}(\sigma) \subseteq \mathbb{A}_n^{\geqslant\lambda^{(1)}}(\sigma) \subseteq \cdots \subseteq \mathbb{A}_n^{\geqslant\lambda^{(m)}}(\sigma) = \mathbb{A}^{\Bbbk}_n(\sigma).$$

where $\mathbb{A}_n^{\geqslant\lambda}(\sigma) = \mathbb{A}^{\Bbbk}_n(\sigma)\langle 1_\nu \mid \nu \in \mathscr{P}^\ell_n, \nu \geqslant \lambda\rangle \mathbb{A}^{\Bbbk}_n(\sigma)$.

Proof. This follows immediately from Propositions 6.4 and 6.7 $\square$

Proposition 6.9. *Let $\lambda, \mu \in \mathscr{C}^\ell_n$ with $(r,c,m) \in [\mu]$, $\imath, \jmath \in (\mathbb{Z}/e\mathbb{Z})^n$, and $A \in {}^{\imath}_{\mu}\mathcal{R}^{\jmath}_{\lambda}$. We have that*

$$y_{(r,c,m)} 1^{\imath}_{\mu} A = A y_{(r',c',m')} 1^{\imath}_{\lambda} + \sum_{\substack{A' \in {}^{\imath}_{\mu}\mathcal{R}^{\jmath}_{\lambda} \\ A' \rhd_\sigma A}} A'.$$

Here $(r',c',m') \in [\lambda]$ on the southern edge is connected to (r,c,m) on the northern edge.

Proof. Now consider a general diagram $A \in 1^{\imath}_{\mu}\mathbb{A}^{\Bbbk}_n(\sigma)1^{\jmath}_{\lambda}$. If there is a dot placed at the top of the strand labelled by $(r,c,m) \in [\mu]$ we move this dot along the strand towards the bottom of the diagram using (homogenous) relations (A2), (A3) and (A12). We hence rewrite A as a linear combination of diagrams $A' \in 1^{\imath}_{\mu}\mathbb{A}^{\Bbbk}_n(\sigma)1^{\jmath}_{\lambda}$ where each A' differs from A only in that one or zero crossings of like-labelled strands have been undone and there is either zero or one dots along the southernmost edge. This amounts to removing zero or one of the Coxeter generators in the reduced expression $[A]$ to obtain a reduced expression of $[A']$. Hence this sum is over diagrams A' such that $A' \rhd_\sigma A$, as required. $\square$

Example 6.10. A step in the procedure outlined in the proof of Proposition 6.9 is carried out in obtaining Figure 4 from Figure 5.

Proposition 6.11. *Let $\lambda, \mu \in \mathscr{C}^\ell_n$ and $\imath, \jmath \in (\mathbb{Z}/e\mathbb{Z})^n$. If $w = s_{t_1,t_1+1} \ldots s_{t_m,t_m+1}$ and $w = s_{r_1,r_1+1} \ldots s_{r_m,r_m+1}$ are two reduced expressions and $A, A' \in {}^{\imath}_{\mu}\mathcal{R}^{\jmath}_{\lambda}$ are two reduced diagrams with $[A] = s_{t_1,t_1+1} \ldots s_{t_m,t_m+1}$ and $[A'] = s_{r_1,r_1+1} \ldots s_{r_m,r_m+1}$, then*

$$A = A' + \sum_{\substack{A'' \in {}^{\imath}_{\mu}\mathcal{R}^{\jmath}_{\lambda} \\ A'' \rhd_\sigma A'}} A'' f_{A''}(y) \in 1^{\imath}_{\mu}\mathbb{A}^{\Bbbk}_n(\sigma)1^{\jmath}_{\lambda} \quad \text{for some } f_{A''}(y) \in \mathcal{Y}^{\jmath}_{\lambda}.$$

Proof. Recall Matsumoto's theorem states that any two reduced expressions for $w \in \mathfrak{S}_n$ differ only by applying a sequence of the relations $s_{i,i+1}s_{i+1,i+2}s_{i,i+1} = s_{i+1,i+2}s_{i,i+1}s_{i+1,i+2}$.

By assumption neither A nor A' contains a double crossing or a dot on any strand. Therefore we can obtain A from A' by applying *only* the strand-through-crossing relations (A8), (A7), and (A10) successively. These are precisely the relations which rewrite a subproduct $s_{i,i+1}s_{i+1,i+2}s_{i,i+1}$ in $[A']$ in the form $s_{i+1,i+2}s_{i,i+1}s_{i+1,i+2} \pm 1_{\mathfrak{S}_{n+2\ell}}$. Thus one can rewrite the diagram in the required form at the expense of some error terms A'' such that $[A''] = u < w$, however we note that A'' may no longer be reduced. If A'' is reduced, then $g_{A''}(y) = \pm 1$. If A'' is not reduced, then rewriting A'' as a linear combination of reduced diagrams involves reapplying some combination of relations (A8), (A7), (A10), or (A5) (which simply creates more error terms A''' with $[A'''] = v \leqslant u < w$ and $g_{A'''}(y) = \pm 1$) and (A6) (which creates error terms A''' with $[A'''] = v < u < w$ and $g_{A'''}(y)$ a polynomial of degree 1). Once this process terminates we are left with a combination of more dominant diagrams, but with dots in the middle of the diagram (which we now need to move southwards). We can isotope the neighbourhood of any diagram (in particular any region containing a dot) so that it is of the form 1_ν for some $\nu \in \mathscr{C}^\ell_n$; we then apply Proposition 6.9 to deduce the result. $\square$

Proposition 6.12. *Let $\lambda, \mu \in \mathscr{C}^\ell_n$ and $\imath, \jmath \in (\mathbb{Z}/e\mathbb{Z})^n$. Let $A \in 1^{\imath}_{\mu}\mathbb{A}^{\Bbbk}_n(\sigma)1^{\jmath}_{\lambda}$ be a diagram which is* not reduced*, then*

$$A = \sum_{\substack{A' \in {}^{\imath}_{\mu}\mathcal{R}^{\jmath}_{\lambda} \\ A' \rhd_\sigma A}} A' f_{A'}(y) \in 1^{\imath}_{\mu}\mathbb{A}^{\Bbbk}_n(\sigma)1^{\jmath}_{\lambda} \quad \text{for some } f_{A''}(y) \in \mathcal{Y}^{\jmath}_{\lambda}.$$

Proof. Any non-reduced diagram $A \in 1^{\imath}_{\mu}\mathbb{A}^{\Bbbk}_{n}(\sigma)1^{\jmath}_{\lambda}$ contains either a double-crossing of strands or a dot on a strand. By Proposition 6.9 we can restrict our attention to the former case. We choose $y \in [\varepsilon, 1-\varepsilon]$ minimal such that $Y = A \cap (\mathbb{R} \times (y, 1])$ contains this double crossing. We have that $X = A \cap (\mathbb{R} \times [0, y))$ does not contain this double-crossing and so X is a reduced diagram. By Proposition 6.11 we can assume that the northern most crossing of strands in X is equal to the crossing of strands in Y modulo a combination of diagrams $X' \rhd_\sigma X$. We have that $A = YX = \sum_{X' \rhd X} YX' = YX + \sum_{A' \rhd A} A'$ where the second term in the sum is of the required form. We are now free to undo the double-crossing in $YX \cap [\mathbb{R} \times (y-\varepsilon, y+\varepsilon)]$ using relation (A5), (A6), (A9). In any case, the result is either 1 or 2 diagrams with this crossing undone (possibly at the expense of acquiring some dots) and so both diagrams are strictly more dominant than A in the Bruhat ordering. By Proposition 6.9 we can remove any dots to obtain a linear combination of diagrams A'' which strictly dominate A and such that $A' \cap (\mathbb{R} \times [0, y])$ is reduced. If each A' is reduced the result follows; if not, then there exists some $1 > y' > y > 0$ for which $A' \cap (\mathbb{R} \times [0, y))$ is not reduced and we can repeat the above argument. Repeating as necessary, the result follows. □

We hence immediately generalise the spanning set of [KL09, Section 2.3] to our algebras. Namely, we can write any element of $\mathbb{A}^{\Bbbk}_{n}(\sigma)$ as a linear combination of reduced diagrams with dots along the southernmost edge.

Corollary 6.13. *The algebra $\mathbb{A}^{\Bbbk}_{n}(\sigma)$ is free as a $\Bbbk$-module and spanned by*

$$\{Af_A(y) \mid \lambda, \mu \in \mathscr{C}^{\ell}_{n}, \imath, \jmath \in (\mathbb{Z}/e\mathbb{Z})^n, A \in {}^{\imath}_{\mu}\mathcal{R}^{\jmath}_{\lambda}, f_A(y) \in \mathcal{Y}^{\jmath}_{\lambda}\}$$

We view the following theorem as the "2-sided" version of Corollary 6.13.

Theorem 6.14. *Let $\Bbbk$ be a integral domain. The $\Bbbk$-algebra $\mathbb{A}^{\Bbbk}_{n}(\sigma)$ is spanned by*

$$\{A_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}_\sigma(\lambda, \mu), \mathsf{T} \in \mathrm{SStd}_\sigma(\lambda, \nu), \lambda \in \mathscr{P}^{\ell}_{n}, \mu, \nu \in \mathscr{C}^{\ell}_{n}\}.$$

Proof. In Theorem 6.8 we saw that any diagram $A \in \mathbb{A}^{\Bbbk}_{n}(\sigma)$ can be written as a linear combination of elements of the form $x_\lambda 1_\lambda y^*_\lambda$ for $x_\lambda, y_\lambda \in \mathbb{A}^{\Bbbk}_{n}(\sigma)1_\lambda$ and $\lambda \in \mathscr{P}^{\ell}_{n}$. It remains to show that the elements x_λ and y_λ can be chosen so that (i) x_λ and y_λ are reduced diagrams (neither has a dot on any strand, or contains any "double-crossing"), (ii) the span of these elements is independent of the choices of reduced expression for $[x_\lambda]$ and $[y_\lambda]$. We shall then conclude that the algebra is spanned by A_{ST} for tableaux S, T of shape $\lambda \in \mathscr{P}^{\ell}_{n}$. Finally it will remain to show that the set of A_{ST} for a pair of *semistandard* tableaux S, T span the algebra.

The result for $\lambda = ((n), \varnothing, \dots, \varnothing)$ is trivial. We assume that (i) is proven for all partitions strictly more dominant than λ. For the remainder of the proof, we refine our induction by proceeding along the Bruhat order on x_λ and y_λ and consider the span of elements modulo

$$\mathrm{Span}_{\Bbbk}\{A_{\mathsf{ST}} \mid A_{\mathsf{S}} \rhd_\sigma x_\lambda \text{ or } A_{\mathsf{T}} \rhd_\sigma y_\lambda\} + \mathbb{A}^{\rhd\lambda}_{n}. \tag{6.1}$$

If x_λ or y_λ has a dot or a double crossing, then $x_\lambda 1_\lambda y^*_\lambda$ is zero modulo equation (6.1) by Proposition 6.9 and Proposition 6.12. Similarly, given any two reduced diagrams x_λ and x'_λ tracing out the same bijection : $[\lambda] \to [\mu]$, we have that $x_\lambda 1_\lambda y_\lambda - x'_\lambda 1_\lambda y_\lambda$ belongs to equation (6.1) by Proposition 6.11. Thus (i) and (ii) hold by induction.

We note that any bijection : $[\lambda] \to [\mu]$ is encoded as a tableau of shape λ and weight μ and so $\{A_{\mathsf{ST}} \mid \mathsf{S}, \mathsf{T}$ are tableaux of shape $\lambda\}$ is a spanning set by definition and our proof of (i) and (ii). It only remains to show that the subset of semistandard tableaux (within the wider class of tableaux) index a spanning set. We consider A_{S} (or A^*_{T}) such that S (or T) violates the semistandardness condition. In other words, one of the following holds: (i) $\mathsf{S}(1,1,m) > s_m$ (ii) $\mathsf{S}(r,c,m) > \mathsf{S}(r-1,c,m)+1$ or (iii) $\mathsf{S}(r,c,m) > \mathsf{S}(r,c-1,m)-1$. In each case, we obtain a "bad crossing". We can choose to draw our diagram A_{S} so that this crossing appears at the bottom of the diagram using Proposition 6.12 and induction on the Bruhat ordering. These crossings are as follows,

(i) the solid strand corresponding to $(1,1,m)$ passes to the right of the red σ_m-strand,
(ii) the ghost strand corresponding to (r,c,m) passes to the right of the solid strand corresponding to $(r-1,c,m)$,

(*iii*) the solid strand corresponding to (r, c, m) passes to the right of the ghost strand corresponding to $(r-1, c, m)$.

In each case the strand labelled by the box (r, c, m) is now free to move right wards using the process outlined in Theorem 6.8 and hence belongs to $\mathbb{A}_n^{\rhd\lambda}(\sigma)$. $\square$

6.3. **The Schur functor.** We define the Schur or KZ functor relating the quiver Hecke and Cherednik algebras. We let $\omega \in \mathscr{P}_n^\ell$ denote the unique element which is minimal in the σ-dominance order. For weakly decreasing $s_1 \geqslant s_2 \geqslant \ldots \geqslant s_\ell$ we have that $\omega = (\varnothing, \varnothing, \ldots, \varnothing, (1^n))$. We let E_ω^σ denote the associated $\mathsf{Schur\ idempotent}$ $\mathsf{E}_\omega^\sigma = \sum_{\imath \in (\mathbb{Z}/e\mathbb{Z})^n} 1_\omega^\imath$.

Definition 6.15. Given $\mathsf{T} \in \mathrm{SStd}_\sigma(\lambda, \omega)$ (respectively $\mathsf{t} \in \mathrm{Std}_\sigma(\lambda)$) we define the reading word $R(\mathsf{T})$ (respectively $r(\mathsf{t})$) to be the ordered sequence of boxes (r_k, c_k, m_k) for $1 \leqslant k \leqslant n$ under the ordering $(r_k, c_k, m_k) < (r_{k'}, c_{k'}, m_{k'})$ if and only if $\mathsf{T}(r_k, c_k, m_k) < \mathsf{T}(r_{k'}, c_{k'}, m_{k'})$ (respectively $\mathsf{t}(r_k, c_k, m_k) < \mathsf{t}(r_{k'}, c_{k'}, m_{k'})$).

Proposition 6.16. *Let $\sigma \in (\mathbb{Z}/e\mathbb{Z})^\ell$ and $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ be a charge. For $\lambda \in \mathscr{P}_n^\ell$, we have a bijection*

$$\varphi : \mathrm{Std}_\sigma(\lambda) \to \mathrm{SStd}_\sigma(\lambda, \omega).$$

given by $\varphi(\mathsf{t}) = \mathsf{T}$ if and only if $r(\mathsf{t}) = R(\mathsf{T})$.

Proof. We order the boxes $(r, 1, \ell)$ for $1 \leqslant r \leqslant n$ of the ℓ-partition $\omega \in \mathscr{P}_n^\ell$ by the natural numbering on $\{1, \ldots, n\}$. Clearly $\mathbf{I}_{(r,1,\ell)}^\sigma < \mathbf{I}_{(r',1,\ell)}^\sigma$ if and only if $1 \leqslant r' < r \leqslant n$. Therefore the set of maps $\{\mathsf{T} \mid \mathsf{T} : [\lambda] \to \mathbf{I}_\omega^\sigma\}$ is in bijection with the set of tableaux of shape λ. This map is simply given by identifying the entry $\mathbf{I}_{(r,1,\ell)}^\sigma \in \mathbb{R}$ in a box of T with the entry $r \in \mathbb{N}$ in a box of t. It remains to show that T is semistandard if and only if t is standard. Condition (*i*) of Definition 1.14 is empty as $\mathbf{I}_{(r,1,\ell)} < s_m$ for all $1 \leqslant m \leqslant \ell$. Conditions (*ii*) and (*iii*) of Definition 1.14 simply correspond to the conditions that $\mathsf{t}(r, c, m) > \mathsf{t}(r-1, c, m)$ and $\mathsf{t}(r, c, m) > \mathsf{t}(r, c-1, m)$ respectively and the fact that $\mathbf{I}_{(r+1,1,\ell)}^\sigma < \mathbf{I}_{(r,1,\ell)}^\sigma - 1$ for $1 \leqslant r \leqslant n$. $\square$

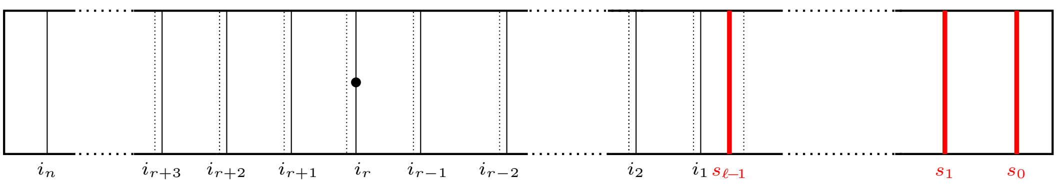
Over a field, the theorem below follows from [Web17, Theorem 4.5] and [Web20, Theorem 5.3]. Our proof proceeds by matching-up the presentations in Definition 3.1 and Definition 8.2 and is valid over a integral domain. By matching up these presentations explicitly, we see how Webster's diagrammatics generalises that of Khovanov–Lauda [KL09]. We also generalise Webster's results to an arbitrary integral domain.

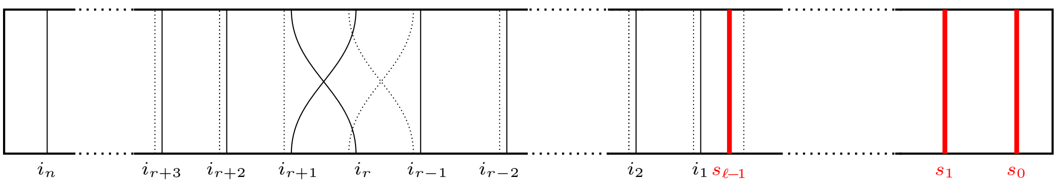
Theorem 6.17. *Let $\Bbbk$ be a integral domain. Let $\underline{s} \in \mathbb{N}_{>1} \times (\mathbb{Z}/e\mathbb{Z})^\ell$ and let $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ be any integral lift. We have an isomorphism of graded $\Bbbk$-algebras*

$$\sigma : \mathcal{H}_n^\Bbbk(\underline{s}) \to \mathsf{E}_\omega^\sigma \mathbb{A}_n^\Bbbk(\sigma) \mathsf{E}_\omega^\sigma$$

which is determined as follows

$\sigma(e(\imath)) = 1_\omega^\imath$

$\sigma(y_r e(\imath)) =$ 

$\sigma(\psi_r e(\imath)) =$ 

Thus we obtain many distinct presentations for the same Hecke algebra, $\mathcal{H}_n^\Bbbk(\underline{s})$, one for each possible lift of $\underline{s} \in (\mathbb{Z}/e\mathbb{Z})^\ell$ to the integers. While these algebras are all isomorphic, we have already seen that each of these distinct lifts has a different combinatorial flavour. We shall see what these different lifts tell us about the structure of $\mathcal{H}_n^\Bbbk(\underline{s})$ in Sections 7 and 9 to 11.

Remark 6.18. The reader might think notice that the diagrams in Theorem 6.17 are only dependent on $\underline{s} \in \mathbb{N}_{>1} \times (\mathbb{Z}/e\mathbb{Z})^\ell$ and not on the integral lift $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$. We remind the reader that this is not the case because the x-coordinates of the red strands are determined by $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ (even though their residue-decorations, $s_{\ell-1}, \dots s_0$, are independent of the integral lift).

Before proving this result, we provide a new set of generators of $\mathsf{E}^\sigma_\omega \mathbb{A}^{\Bbbk}_n(\sigma)\mathsf{E}^\sigma_\omega$ which is highly compatible with the desired isomorphism.

Proposition 6.19. *The algebra $\mathsf{E}^\sigma_\omega \mathbb{A}^{\Bbbk}_n(\sigma)\mathsf{E}^\sigma_\omega$ is generated by*

$$\langle 1^\imath_\omega, \sigma(y_s(e(\imath))), \sigma(\psi_r(e(\imath))) \mid \ 1 \leqslant r < n,\ 1 \leqslant s \leqslant n,\ \textit{and}\ \imath \in (\mathbb{Z}/e\mathbb{Z})^\ell \ \rangle \tag{6.2}$$

subject to (A1) to (A13). In particular, the $\Bbbk$-linear map $\sigma : \mathcal{H}^{\Bbbk}_n(\underline{s}) \to \mathsf{E}^\sigma_\omega \mathbb{A}^{\Bbbk}_n(\sigma)\mathsf{E}^\sigma_\omega$ is surjective.

Proof. We begin by rewriting every diagram in $\mathsf{E}^\sigma_\omega \mathbb{A}^{\Bbbk}_n(\sigma)\mathsf{E}^\sigma_\omega$ so that the red strands are "replaced" with something more akin to the usual cyclotomic relation for the classical KLR-algebra.

Claim: Let $A \in 1^\imath_\omega \mathbb{A}^{\Bbbk}_n(\sigma) 1^\jmath_\omega$ be an arbitrary diagram. Let $\overline{A}$ be the diagram obtained from A by the following procedure:

(1) pull each red strands sufficiently far to the right that it longer intersects any solid or ghost strands;
(2) for each crossing involving a solid i-strand (from southwest to northeast) and a red-strand (from southeast to northwest) place a dot at the position of the crossing in A.

We claim that $A = \overline{A}$.

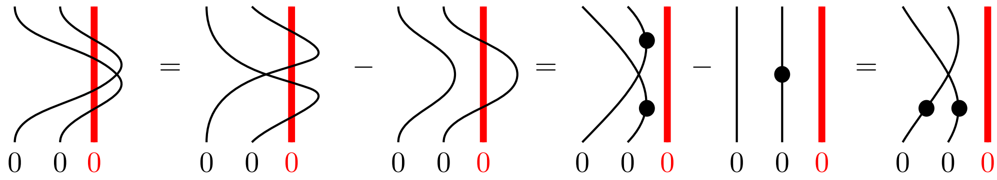


FIGURE 17. Passing a crossing through a red strand using relation (A10), then relation (A9), then (A6)). All ghost strands commute with red strands, and so we do not picture these here.

We now prove the claim. We consider the effect of pulling the mth red strand to the right. We can restrict our attention to the effect on the sub-diagram of $A_{\underline{w}}$ (with underlying word, $[A]$, equal to $\underline{w}$) consisting only of the solid strands of residue $s_m \in \mathbb{Z}/e\mathbb{Z}$ which cross the mth red strand (we can ignore their ghosts, by (A11)). We can further restrict our attention to only consider the crossings of these strands within the region $(\sigma_m - m/\ell, \infty) \times [0,1]$ as it is only these strands within this region to which we need apply non-trivial relations (while pulling the mth red strand to the right). Thus we write $A_{\underline{w}} = A_{\underline{u}} A_{\underline{w}'} A_{\underline{v}}$ where $A_{\underline{w}'}$ consists of the s_m-strands intersected with the region $(\sigma_m - m/\ell, \infty) \times [y_1, y_2]$ (where y_1 and y_2 are chosen appropriately) and $A_{\underline{u}}$ and $A_{\underline{v}}$ consist of the s_m-strands with the regions $(-\infty, \sigma_m - m/\ell,] \times [y_2, 1]$ and $(-\infty, \sigma_m - m/\ell,] \times [0, y_1]$ respectively. We set k to be the number of solid i-strands in $A_{\underline{w}'}$. We proceed by induction on $\ell(\underline{w}')$.

We have that $\ell(\underline{w}') \geqslant 2k$ because each solid strand intersects the red strand at least twice. If $\ell(\underline{w}') = 2k$ then the claim follows by k applications of the leftmost relation in (A9). We can now suppose $\underline{w}' = \underline{x} s_{1,2} \underline{y}$ (if $s_{1,2}$ does not appear in $\underline{w}'$ then we can apply the leftmost relation of (A9)) for $\underline{x}$ and $\underline{y}$ subwords such that $\ell(\underline{x}) + \ell(\underline{y}) < \ell(\underline{w})$. We pull the red-strand through the crossing corresponding to this $s_{1,2}$ at the expense of an error term (with coefficient -1) in which we undo this crossing. By our inductive assumption we can move the red strand through $A_{\underline{xy}}$ and obtain $\overline{A}_{\underline{x}} A_{s_{1,2}} \overline{A}_{\underline{y}} - \overline{A}_{\underline{xy}}$ but with the red strand all the way to the right. By construction, we have a dot at the bottom of $A'_{\underline{x}}$ on the leftmost strand; using (A6) we pull this dot down to the left through $A_{s_{1,2}}$ to obtain the required diagram $\overline{A}_{\underline{w}'}$ at the expense of an error term equal to $\overline{A}_{\underline{xy}}$ (with coefficient $+1$). The error terms cancel and the claim holds. An example is depicted in Figure 17 (for $\underline{w} = \underline{w}' = s_{1,2}$).

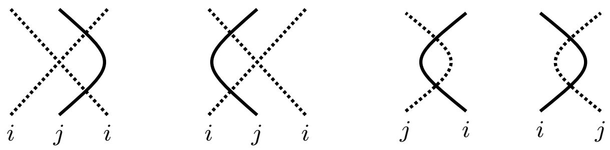


FIGURE 18. Crossings within a diagram $A \in \mathsf{E}^\sigma_\omega \mathbb{A}^{\Bbbk}_n(\sigma)\mathsf{E}^\sigma_\omega$.

With the claim in place, we may now assume $A \in \mathsf{E}^\sigma_\omega \mathbb{A}^{\Bbbk}_n(\sigma)\mathsf{E}^\sigma_\omega$ has no crossings involving red strands. Any crossing in such a diagram is of one of the forms depicted in Figure 18.

We can undo any double-crossing using relation (A4) and (A5) to obtain a sum of diagrams of the required form. We must now consider any triple-crossings as in Figure 18 and show that this belongs to a wider product of the form

$$\sigma(\psi_{r+1}(e(\imath)))\sigma(\psi_r(e(\imath)))\sigma(\psi_{r+1}(e(\imath))) \qquad \text{or} \qquad \sigma(\psi_r(e(\imath)))\sigma(\psi_{r+1}(e(\imath)))\sigma(\psi_r(e(\imath))) \tag{6.3}$$

for $1 \leqslant r < n$. There are precisely 4 different wider products (up to isotopy) to which such a diagram can belong, these are depicted in Figure 19. The first and fourth of these diagrams are already of the respective forms in equation (6.3).

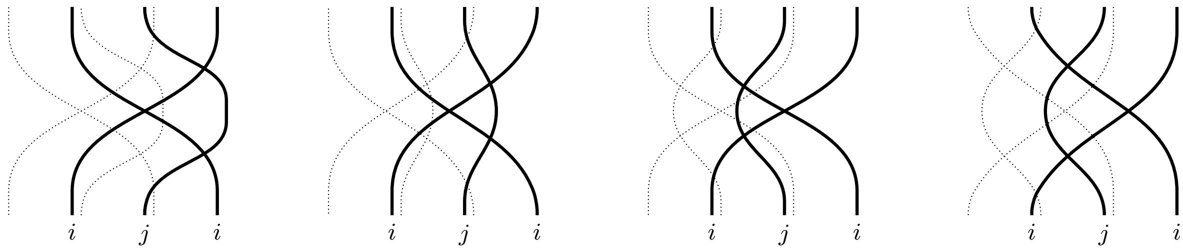


FIGURE 19. The wider diagrams $A \in \mathsf{E}^\sigma_\omega \mathbb{A}^{\Bbbk}_n(\sigma)\mathsf{E}^\sigma_\omega$ containing a triple-crossing of the form depicted in Figure 18. The first and final of which are already of the required form.

For the remaining two diagrams in Figure 19, we can pull the ghost j-strand (respectively, solid j-strand) to the right (to the left) of the solid (respectively ghost) i-crossing using (A6) or (A5). In either case, the resulting diagram is now of the required form. □

Proof of Theorem 6.14. We have already seen that $\mathsf{E}^\sigma_\omega \mathbb{A}^{\Bbbk}_n(\sigma)\mathsf{E}^\sigma_\omega$ is generated by the elements $1^\imath_\omega$, $\sigma(y_s(e(\imath)))$, $\sigma(\psi_r(e(\imath)))$ for $1 \leqslant s \leqslant n$ and $1 \leqslant r < n$ and $\imath \in (\mathbb{Z}/e\mathbb{Z})^n$. We define σ^{-1} to be the obvious inverse map. We will now verify that σ respects 3.2 to 3.13 and that σ^{-1} respects relations (A1) to (A13).

The images of relations 3.2 to 3.8 under σ follow from the diagrammatic definition of the multiplication and (A1). Conversely, the image under σ^{-1} of the implicit diagrammatic relations (i.e. that strands carry residues, products are zero for non-matching residue sequences, and (A1)) immediately follow from 3.2 to 3.8. The images of relations 3.9 to 3.10 under σ follow from (A2) and (A3). Conversely the image of relations (A2) and (A3) under σ^{-1} follow from 3.9 and 3.10.

We shall now show that the image of 3.11 under σ holds in $\mathsf{E}^\sigma_\omega \mathbb{A}^{\Bbbk}_n(\sigma)\mathsf{E}^\sigma_\omega$ and that the images of (A4), (A5) and (A6) under σ^{-1} hold in $\mathcal{H}^{\Bbbk}_n(\underline{s})$. Relation 3.11 has four parts; the images of the $i_r \neq i_{r+1} \pm 1$ cases follow from relation (A4) and (A5). If $i_r - 1 = i_{r+1}$, then we first apply relation (A6) to the diagram $\sigma(\psi_r e(\imath))\sigma(\psi_r e(s_{r,r+1}\imath))$ in order to undo the double-crossing of the ghost $(i-1)$-strand with the solid i-strand; now if $e \neq 2$, then this implies that $i_r \neq i_{r+1} - 1$ and so the double-crossing of the ghost i-strand with the solid $(i-1)$-strand can be undone without cost by relation (A5). We hence obtain that $\sigma(\psi_r e(\imath))\sigma(\psi_r e(s_{r,r+1}\imath)) = \sigma(y_{r+1}e(\imath)) - \sigma(y_r e(\imath))$, as required. We now assume that $i_r + 1 = i_{r+1}$ with $e \neq 2$. Here we have that the double-crossing of the ghost i-strand with the solid $(i+1)$-strand can be undone without cost by relation (A5); now the double-crossing of the ghost $(i-1)$-strand with the solid i-strand can be undone by relation (A6) to obtain $\sigma(\psi_r e(\imath))\sigma(\psi_r e(s_{r,r+1}\imath)) = \sigma(y_r e(\imath)) - \sigma(y_{r+1}e(\imath))$, as required. Finally the $e = 2$

case can be obtained in the same fashion as above, except noting that $i_r = i_{r+1}+1 = i_{r+1}-1$ and so we need apply relation (A6) twice and hence obtain four terms.

Conversely, let $A \in \mathsf{E}^\sigma_\omega \mathbb{A}^\Bbbk_n(\sigma)\mathsf{E}^\sigma_\omega$ be any diagram written as a product of the generators in equation (6.2). Any double crossing in A of the form depicted in (A4), (A5), (A6) must occur within a region of A of the form

$$\sigma(\psi_r)\sigma(\psi_r)1_\omega^{(\dots,i_{r-1},i_r,\dots)}. \tag{6.4}$$

In particular, none of the double-crossings in (A4), (A5), (A6) ever appear by themselves; they always appear with a complementary pair of double-crossings strands (depending on $i_{r-1} = i_r - 1, i_r, i_r+1$ or otherwise). Thus we do not need to show that the images under σ^{-1} of (A4), (A5), (A6) themselves hold, but rather we need only check that all possible pairs of these relations (which can appear in equation (6.4)) hold. These pairs correspond precisely to the subcases of 3.11 and can be argued identically to the above (but backwards).

Now we consider relation 3.12 at the same time as (A7) and (A8). We first check that σ respects 3.12 for each of the four cases. In the first case of 3.12, we can move the ghost $(i-1)$-strand in $\sigma(\psi_{r+1}\psi_r\psi_{r+1}e(\dots,i,i-1,i\dots))$ through the solid i-crossing using (A7) at the expense of an error term (with coefficient -1) in which we undo the solid i-crossing; we can then simplify the former diagram using (A8) and the error diagram using (A5) in order to obtain $\sigma((\psi_r\psi_{r+1}\psi_r - 1)e(\dots,i,i+1,i\dots))$. Figure 19 depicts the three steps in this process, with only the step from the first to the second diagram producing an error term (i.e. we can get from the second to the third to the fourth diagram in Figure 19 using only (A8)). The second case is similar. The third case is an amalgamation of cases 1 and 2, but the error terms must be simplified with (A6) due to the additional residue adjacencies. The fourth case follows directly from relation (A8). Thus σ preserves relation 3.12.

Conversely, by Proposition 6.19 any triple crossing of the form in (A7) or (A8) must occur within a region of the diagram of the form

$$\sigma(\psi_r)\sigma(\psi_{r+1})\sigma(\psi_r)1_\omega^{(\dots,i_{r-1},i_r,i_{r+1}\dots)} \qquad \text{or} \qquad \sigma(\psi_{r+1})\sigma(\psi_r)\sigma(\psi_{r+1})1_\omega^{(\dots,i_{r-1},i_r,i_{r+1}\dots)} \tag{6.5}$$

for some $i_{r-1}, i_r, i_{r+1} \in \mathbb{Z}/e\mathbb{Z}$. In particular, none of the triple-crossings in (A7) or (A8) ever appear by themselves; they always appear with a complementary triple-crossing strands (breaking down into cases according to whether $i_{r-1} = i_r - 1, i_r, i_r+1$ or otherwise). Thus we do not need to show that the images under σ^{-1} of (A7) or (A8) themselves hold, but rather we need only check that the possible combinations of these relations (which can appear in equation (6.5)) hold. These pairs correspond precisely to the subcases of 3.12 and can be argued identically to the above (but backwards).

Finally it remains to consider relation 3.13 for $\mathcal{H}^\Bbbk_n(\underline{s})$ at the same time as the red strand relations in $\mathsf{E}^\sigma_\omega \mathbb{A}^\Bbbk_n(\sigma)\mathsf{E}^\sigma_\omega$. We have that

$$\sigma(y_1^{\sharp\{s_m \mid s_m = i_1\}}e(\imath)) = 0$$

by ℓ applications of (A9). We now consider the image under σ^{-1} of the relations involving red strands. We have shown in the proof of Proposition 6.19 that any diagram $A \in \mathsf{E}^\sigma_\omega \mathbb{A}^\Bbbk_n(\sigma)\mathsf{E}^\sigma_\omega$ is equal to some (decorated) diagram $\overline{A}$ with no crossings involving red strands. Thus we need only to verify that if A is unsteady, then $\sigma^{-1}(\overline{A}) = 0$. If A is unsteady, then we can move the strands back towards the left, with the rightmost solid strand (of residue $i \in \mathbb{Z}/e\mathbb{Z}$, say) picking up a total of $\sharp\{s_m \mid s_m = i\}$ dots; the resulting diagram factors through an idempotent of the form $y_{(1,1,\ell)}^{\sharp\{s_m|s_m=i_1\}}1_\omega^\imath = \sigma(y_1^{\sharp\{s_m|s_m=i_1\}}e(\imath))$ and thus $\sigma^{-1}(A) = 0$ by 3.13, as required. □

6.4. **Cellularity and quasi-heredity of quiver Cherednik algebras.** We shall now show that the spanning set of Theorem 6.14 is in fact a cellular basis of the quiver Cherednik algebra. We first require a new ordering on the boxes in an ℓ-multipartition.

Definition 6.20. Given $\mathsf{S} \in \mathrm{SStd}_\sigma(\lambda,\mu)$ and two distinct boxes $(r,c,m),(r',c',m') \in \lambda$ we write $(r,c,m) \succ (r',c',m')$ if one of the following holds

(*i*) $\mathbf{I}^\sigma_{(r,c,m)} > \mathbf{I}^\sigma_{(r',c',m')} \pm 1$ and $\mathsf{S}(r,c,m) < \mathsf{S}(r',c',m') \pm 1$ or
(*ii*) $\mathbf{I}^\sigma_{(r,c,m)} > \mathbf{I}^\sigma_{(r',c',m')}$ and $\mathsf{S}(r,c,m) < \mathsf{S}(r',c',m')$ or

(*iii*) (r,c,m) and (r',c',m') appear in the same row (respectively column) of the same component of λ and $c = c'+1$ (respectively $r = r'+1$).

We write $(r,c,m) \succeq (r',c',m')$ if either $(r,c,m) = (r',c',m')$ or $(r,c,m) \succeq (r',c',m')$.

Definition 6.21. Given $\mathsf{S} \in \mathrm{SStd}_\sigma(\lambda,\mu)$ and $\mathsf{t} \in \mathrm{Std}_\sigma(\lambda)$, we say that that t factors through S if $(r,c,m) \succ (r',c',m')$ implies $\mathsf{t}(r,c,m) > \mathsf{t}(r',c',m')$ for all pairs of distinct $(r,c,m),(r',c',m') \in \lambda$.

Proposition 6.22. *Given $\mathsf{S} \in \mathrm{SStd}_\sigma(\lambda,\mu)$ there exists an $\bar{\mathsf{s}} \in \mathrm{Std}_\sigma(\lambda)$ such that $\bar{\mathsf{s}}$ factors through S. For such a pair, $(\mathsf{S},\bar{\mathsf{s}})$, there exists a tableau S^c of shape μ and weight ω such that $A_{\mathsf{S}^c}A_{\mathsf{S}} = A_{\varphi(\bar{\mathsf{s}})}$.*

Proof. Let $\mathsf{S} \in \mathrm{SStd}_\sigma(\lambda,\mu)$. Let $(r_1,c_1,m_1),(r_2,c_2,m_2) \in \lambda$ be any pair of distinct boxes such that $(r_1,c_1,m_1) \succ (r_1,c_2,m_2)$. By Definition 1.14, if $(r_1,c_1,m_1),(r_2,c_2,m_2)$ are as in (*iii*) then

$$|\mathsf{S}(r_1,c_1,m_1) - \mathbf{I}^\sigma_{(r_1,c_1,m_1)}| \geqslant |\mathsf{S}(r_2,c_2,m_2) - \mathbf{I}^\sigma_{(r_2,c_2,m_2)}|$$

and if $(r_1,c_1,m_1),(r_1,c_2,m_2)$ are as in (*i*) or (*ii*) then

$$|\mathsf{S}(r_1,c_1,m_1) - \mathbf{I}^\sigma_{(r_1,c_1,m_1)}| > |\mathsf{S}(r_2,c_2,m_2) - \mathbf{I}^\sigma_{(r_2,c_2,m_2)}|.$$

Therefore if $(r_1,c_1,m_1) \succ (r_2,c_2,m_2) \succ \cdots \succ (r_k,c_k,m_k)$ then

$$|\mathsf{S}(r_1,c_1,m_1) - \mathbf{I}^\sigma_{(r_1,c_1,m_1)}| \geqslant |\mathsf{S}(r_k,c_k,m_k) - \mathbf{I}^\sigma_{(r_k,c_k,m_k)}| \tag{6.6}$$

with equality only if $m_1 = m_k$ and $r_1 \geqslant r_k$, $c_1 \geqslant c_k$ (and there are no crossings between the strands labelled by these boxes).

We consider the transitive closure of the relation $\succeq$ (by abuse of notation we also denote this by $\succeq$); this relation is transitive and reflexive by definition. If $(r,c,m) \succeq (r',c',m')$ and $(r,c,m) \preceq (r',c',m')$ then by equation (6.6) we have that $(r,c,m) = (r',c',m')$; hence the relation is antisymmetric. Therefore $\succeq$ defines a partial ordering on the boxes of $\lambda \in \mathscr{P}^\ell_n$.

Regard $\succ$ as a partial ordering on the boxes of $\mathsf{S}(\lambda) = \mu$ by identifying the nodes of μ with the corresponding nodes of λ. We can encode any total refinement, $\succ_{\mathrm{t}}$, of $\succ$ as a tableau, S^c, of shape μ and weight ω. This is simply given by letting $\mathsf{S}^c(r,c,m) < \mathsf{S}^c(r',c',m')$ if and only if $(r,c,m) \succ_{\mathrm{t}} (r',c',m')$ for $(r,c,m),(r',c',m') \in \mu$.

It remains to show that $A_{\mathsf{S}^c}A_{\mathsf{S}} = A_{\varphi(\bar{\mathsf{s}})}$ for some $\bar{\mathsf{s}} \in \mathrm{Std}_\sigma(\lambda)$. Suppose (r,c,m) and (r',c',m') are two boxes in λ whose solid or ghost strands cross in the diagram A_{S}. In which case, $(r,c,m) \succ (r',c',m')$ (or vice versa) and we are as in one of cases (*i*), (*ii*), or (*iii*) of Definition 6.20. By definition, $\mathsf{S}^c(r,c,m) < \mathsf{S}^c(r',c',m')$ and so the crossing strands from A_{S} do not cross again in A_{S^c}. Therefore the diagram $A_{\mathsf{S}^c}A_{\mathsf{S}}$ contains no double-crossings and so is equal to $A_{\overline{\mathsf{S}}}$ for $\overline{\mathsf{S}}$ some tableau of shape λ and weight ω. Now, by condition (*iii*) of Definition 6.20, we have that $\overline{\mathsf{S}}(r,c+1,m) < \overline{\mathsf{S}}(r,c,m)$ and $\overline{\mathsf{S}}(r+1,c,m) < \overline{\mathsf{S}}(r,c,m)$ for all $1 \leqslant r,c \leqslant n$ and $1 \leqslant m \leqslant \ell$. Since any pair of boxes of the ℓ-partition ω are at 1 unit apart, $\overline{\mathsf{S}}$ satisfies conditions (*i*) and (*ii*) of Definition 1.14. Finally for any $1 \leqslant m \leqslant \ell$, we have that $\overline{\mathsf{S}}(1,1,m) = (r,1,\ell)$ for some $1 \leqslant r \leqslant n$ and so $\overline{\mathsf{S}}$ satisfies condition (*iii*) of Definition 1.14. Therefore $\overline{\mathsf{S}}$ is semistandard. Finally, we let $\bar{\mathsf{s}}$ be the standard tableau determined by $\varphi(\bar{\mathsf{s}}) = \overline{\mathsf{S}}$ and this completes the proof. □

Finally, we generalise [Web17, Theorem 4.11] to an arbitrary integral domain.

Theorem 6.23. *Let $\Bbbk$ be an arbitrary integral domain. The algebra $\mathbb{A}^{\Bbbk}_n(\sigma)$ is free as an $\Bbbk$-module and has a graded cellular basis*

$$\{A_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}_\sigma(\lambda,\mu), \mathsf{T} \in \mathrm{SStd}_\sigma(\lambda,\nu), \lambda \in \mathscr{P}^\ell_n, \mu,\nu \in \mathscr{C}^\ell_n\}$$

with respect to the σ-dominance order on $\mathscr{P}^\ell_n$ and the involution $*$ given by horizontal reflection. We let $\Delta^{\Bbbk}_\sigma(\lambda)$ denote the corresponding cell-module for $\lambda \in \mathscr{P}^\ell_n$.

Proof. We shall prove this by contradiction. By Theorem 6.8 and the fact that $\sum_{\alpha \in \mathscr{C}^\ell_n, \imath \in (\mathbb{Z}/e\mathbb{Z})^n} 1^\imath_\alpha$ is the identity of $\mathbb{A}^{\Bbbk}_n(\sigma)$, it is enough to show that if there exist $\alpha_{\mathsf{UV}} \in \Bbbk$ such that

$$\sum_{\substack{\mathsf{U} \in \mathrm{SStd}_\sigma(\lambda,\mu) \\ \mathsf{V} \in \mathrm{SStd}_\sigma(\lambda,\nu)}} \alpha_{\mathsf{UV}} A_{\mathsf{UV}} = 0 \mod \mathbb{A}^{\rhd\lambda}_n(\sigma) \tag{6.7}$$

then this implies that $\alpha_{\mathsf{UV}} = 0$ for all $\mathsf{U} \in \mathrm{SStd}_\sigma(\lambda,\mu), \mathsf{V} \in \mathrm{SStd}_\sigma(\lambda,\nu)$. We set $\sharp(\mathsf{S},\mathsf{T}) = \ell[A_\mathsf{S}] + \ell[A_\mathsf{T}]$. We let S,T be any pair such that $\sharp(\mathsf{S},\mathsf{T}) \geqslant \sharp(\mathsf{U},\mathsf{V})$ for any pair of tableaux U,V with $\alpha_{\mathsf{UV}} \neq 0$. We let S^c (respectively T^c) denote any tableau of shape μ (respectively ν) and shape ω as in Proposition 6.22. We shall show that the coefficient α_{ST} is necessarily zero (and so the result immediately follows by repeating this argument). We multiply equation (6.7) on the left by $A^*_{\mathsf{S}^c}$ and on the right by A_{T^c}; it is enough to show that if

$$\alpha_{\mathsf{ST}} A_{\overline{\mathsf{S}}\,\overline{\mathsf{T}}} + \sum_{\substack{\mathsf{U}\in\mathrm{SStd}_\sigma(\lambda,\mu)\\ \mathsf{V}\in\mathrm{SStd}_\sigma(\lambda,\nu)}} \alpha_{\mathsf{UV}} A_{\mathsf{S}^c} A_{\mathsf{UV}} A^*_{\mathsf{T}^c} = 0 \mod \mathbb{A}_n^{\rhd\lambda}(\sigma), \tag{6.8}$$

then $\alpha_{\mathsf{ST}} = 0$ (where $\overline{\mathsf{S}} = \varphi(\bar{\mathsf{s}})$ and $\overline{\mathsf{T}} = \varphi(\bar{\mathsf{t}})$ as in Proposition 6.22). There are two cases to consider. Firstly, if one of $A_{\mathsf{S}^c}A_\mathsf{U}$ or $A^*_\mathsf{V} A^*_{\mathsf{T}^c}$ contains a double-crossing, then

$$A_{\mathsf{S}^c} A_{\mathsf{UV}} A^*_{\mathsf{T}^c} = \sum_{\substack{\mathsf{U}',\mathsf{V}'\in\mathrm{SStd}_\sigma(\lambda,\omega)\\ \sharp(\mathsf{U}',\mathsf{V}')<\sharp(\mathsf{T},\mathsf{V})}} \beta_{\mathsf{U}'\mathsf{V}'} A_{\mathsf{U}'\mathsf{V}'} \mod \mathbb{A}_n^{\rhd\lambda}(\sigma)$$

for some $\beta_{\mathsf{U}'\mathsf{V}'} \in \Bbbk$, by Proposition 6.12. We now consider the case in which $A_{\mathsf{S}^c}A_\mathsf{U}$ and $A^*_\mathsf{V} A^*_{\mathsf{T}^c}$ contain no double-crossings. We have that $(\mathsf{S},\mathsf{T}) \neq (\mathsf{U},\mathsf{V})$. Assume $\mathsf{U} \neq \mathsf{S}$, then the bijection traced out by U is different to that traced out by S; therefore the bijection traced out by $A_{\mathsf{S}^c}A_\mathsf{U}$ is not equal to that traced out by $A_{\mathsf{S}^c}A_\mathsf{S} = A_{\varphi(\bar{\mathsf{s}})}$. In particular, if $A_{\mathsf{S}^c}A_\mathsf{U}$ contains no double-crossings, then it is equal to $A_{\overline{\mathsf{U}}}$ for $\overline{\mathsf{U}}$ some (not necessarily semistandard) tableau of shape λ and weight ω which is *not* equal to $\varphi(\bar{\mathsf{s}})$. Arguing similarly for the case $\mathsf{V} \neq \mathsf{T}$, we therefore deduce that

$$A_{\mathsf{S}^c} A_{\mathsf{UV}} A^*_{\mathsf{T}^c} = A_{\overline{\mathsf{U}}\,\overline{\mathsf{V}}}$$

for $\overline{\mathsf{U}},\overline{\mathsf{V}}$ two (not necessarily semistandard) tableaux of shape λ such that $(\overline{\mathsf{U}},\overline{\mathsf{V}}) \neq (\varphi(\mathsf{s}),\varphi(\mathsf{t}))$. Now, if $\overline{\mathsf{U}}$ and $\overline{\mathsf{V}}$ are not semistandard, then

$$A_{\overline{\mathsf{U}}\,\overline{\mathsf{V}}} = \sum_{\substack{\mathsf{U}',\mathsf{V}'\in\mathrm{SStd}_\sigma(\lambda,\omega)\\ \sharp(\mathsf{U}',\mathsf{V}')<\sharp(\mathsf{S},\mathsf{T})}} \gamma_{\mathsf{U}'\mathsf{V}'} A_{\mathsf{U}'\mathsf{V}'} \mod \mathbb{A}_n^{\rhd\lambda}(\sigma)$$

for some $\gamma_{\mathsf{U}'\mathsf{V}'} \in \Bbbk$, by as in the proof of Theorem 6.14. If $\overline{\mathsf{U}}$ and $\overline{\mathsf{V}}$ are semistandard, then we set $\overline{\mathsf{U}} = \mathsf{U}'$ and $\overline{\mathsf{V}} = \mathsf{V}'$ for convenience. Putting all of this together, we have that equation (6.8) is equivalent to

$$\alpha_{\mathsf{ST}} A_{\overline{\mathsf{S}}\,\overline{\mathsf{T}}} + \sum_{\substack{\mathsf{U}',\mathsf{V}'\in\mathrm{SStd}_\sigma(\lambda,\omega)\\ (\mathsf{U}',\mathsf{V}')\neq(\mathsf{S},\mathsf{T})}} \alpha_{\mathsf{UV}}(\beta_{\mathsf{U}'\mathsf{V}'} + \gamma_{\mathsf{U}'\mathsf{V}'}) A_{\mathsf{U}'\mathsf{V}'} = 0 \mod \mathbb{A}_n^{\rhd\lambda}(\sigma).$$

Now, the set $\{A_{\mathsf{QR}} \mid \mathsf{Q},\mathsf{R} \in \mathrm{SStd}_\sigma(\lambda,\omega)\}$ is a basis of $\mathsf{E}^\sigma_\omega \mathbb{A}^\Bbbk_n(\sigma)\mathsf{E}^\sigma_\omega$ by Theorem 6.17 and so $\alpha_{\mathsf{ST}} = 0$, as required. Therefore we have verified condition (2) of Definition 2.2. Conditions (1) and (4) of Definition 2.2 follow immediately from the diagrammatic definitions. Condition (3) follows from Propositions 6.9 and 6.12. □

Corollary 6.24 ([Web17, Cor 2.26])**.** *Let $\Bbbk$ be a field. The algebra $\mathbb{A}^\Bbbk_n(\sigma)$ is quasi-hereditary and the $L^\Bbbk_\sigma(\lambda) = \Delta^\Bbbk_\sigma(\lambda)/\mathrm{rad}(\Delta^\Bbbk_\sigma(\lambda))$ for $\lambda \in \mathscr{P}^\ell_n$ provide a complete set of non-isomorphic irreducible modules.*

Proof. Let T denote the unique element of $\mathrm{SStd}_\sigma(\lambda,\lambda)$. The element $A_{\mathsf{TT}} = 1_\lambda \in \mathbb{A}_n^{\trianglerighteq\lambda}(\sigma)$ is an idempotent. Therefore the radical of the bilinear form is not the whole cell module. Therefore the algebra is quasi-hereditary with the prescribed set of irreducible modules. □

7. The many integral cellular bases of quiver Hecke algebras

We now proceed to apply the many Schur functors in order to obtain our many graded cellular bases of Hecke algebras. Given $\underline{s} = (e; s_0, s_2, \dots, s_{\ell-1}) \in \mathbb{N}_{>1} \times (\mathbb{Z}/e\mathbb{Z})^\ell$ we let $\sigma = (e; \sigma_0, \sigma_1, \dots, \sigma_{\ell-1}) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ denote any choice of integral lift. We have seen that $\mathcal{H}^\Bbbk_n(\underline{s}) \cong \mathsf{E}^\sigma_\omega \mathbb{A}^\Bbbk_n(\sigma)\mathsf{E}^\sigma_\omega$ is generated by

$$\langle \sigma(y_1), \dots, \sigma(y_n), \sigma(\psi_1), \dots, \sigma(\psi_{n-1}), \sigma(e(\imath)) \mid \imath \in (\mathbb{Z}/e\mathbb{Z})^n \rangle$$

subject to relations (A1) to (A13). This idea should be very familiar to those working with Cherednik algebras. Given a fixed Hecke algebra $\mathcal{H}_n^{\Bbbk}(\underline{s})$ there are many associated quiver Cherednik algebras $\mathbb{A}_n^{\Bbbk}(\sigma)$ (namely, one for each integral lift $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$). Each of these distinct quiver Cherednik algebras casts its own "charged shadow" on the representation theory of our fixed Hecke algebra. Given $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_\sigma(\lambda)$ we set

$$A^\sigma_{\mathsf{st}} := \mathsf{E}^\sigma_\omega A_{\mathsf{ST}} \mathsf{E}^\sigma_\omega \in \mathcal{H}_n^{\Bbbk}(\underline{s})$$

where $\varphi(\mathsf{s}) = \mathsf{S} \in \mathrm{SStd}_\sigma(\lambda, \omega)$ and $\varphi(\mathsf{t}) = \mathsf{T} \in \mathrm{SStd}_\sigma(\lambda, \omega)$.

Theorem 7.1. *For a charge $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$, the $\Bbbk$-algebra $\mathcal{H}_n^{\Bbbk}(\underline{s})$ admits a graded cellular structure with respect to the poset $(\mathscr{P}_n^\ell, \rhd_\sigma)$ and the basis*

$$\{A^\sigma_{\mathsf{st}} \mid \lambda \in \mathscr{P}_n^\ell, \mathsf{s}, \mathsf{t} \in \mathrm{Std}_\sigma(\lambda)\}$$

and the involution $*$. In particular, $\deg(A^\sigma_{\mathsf{st}}) = \deg(\mathsf{s}) + \deg(\mathsf{t})$ *for* $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_\sigma(\lambda)$.

Proof. The elements $\mathsf{E}^\sigma_\omega A_{\mathsf{ST}} \mathsf{E}^\sigma_\omega = A^\sigma_{\mathsf{ST}}$ satisfy property (2) for $\mathsf{E}^\sigma_\omega \mathbb{A}_n^{\Bbbk}(\sigma) \mathsf{E}^\sigma_\omega \cong \mathcal{H}_n^{\Bbbk}(\underline{s})$ and property (4) immediately. We have that

$$(\mathsf{E}^\sigma_\omega A \mathsf{E}^\sigma_\omega)(\mathsf{E}^\sigma_\omega A_{\mathsf{ST}} \mathsf{E}^\sigma_\omega) = (\mathsf{E}^\sigma_\omega A)(A_{\mathsf{ST}})$$

for $\mathsf{S}, \mathsf{T} \in \mathrm{SStd}_\sigma(\lambda, \omega)$ and therefore (3) for $\mathsf{E}^\sigma_\omega \mathbb{A}_n^{\Bbbk}(\sigma) \mathsf{E}^\sigma_\omega$ follows from condition (3) for $\mathbb{A}_n^{\Bbbk}(\sigma)$. To see that condition (1) holds, we proceed by induction on $n \in \mathbb{N}$. The $n = 0$ case holds trivially. Now, let $\mathsf{s} \in \mathrm{Std}_\sigma(\lambda)$ and let $(r, c, m) \in \lambda$ be such that $\mathsf{s}(r, c, m) = n$. By induction, we may assume that

$$\deg(\mathsf{s}{\downarrow}_{\{1,\dots,n-1\}}) = \deg(A^\sigma_{\mathsf{s}{\downarrow}_{\{1,\dots,n-1\}}})$$

having trivially verified that the $n = 1$ case holds. Now, we have that

$$A^\sigma_{\mathsf{s}} = \overline{A}^\sigma_{\mathsf{s}{\downarrow}_{\{1,\dots,n-1\}}} \times 1^{\lambda+(n,1,\ell)}_{\lambda+(r,c,m)}$$

where the diagrams on the righthand-side are constructed as follows

- we obtain $\overline{A}^\sigma_{\mathsf{s}{\downarrow}_{\{1,\dots,n-1\}}}$ from $A^\sigma_{\mathsf{s}{\downarrow}_{\{1,\dots,n-1\}}}$ by adding a vertical solid strand with x-coordinate $\mathbf{I}^\sigma_{(n,1,\ell)}$;
- we obtain $1^{\lambda+(n,1,\ell)}_{\lambda+(r,c,m)}$ from 1_λ by adding a solid strand, S_n, from $(\mathbf{I}^\sigma_{(r,c,m)}, 0)$ to $(\mathbf{I}^\sigma_{(n,1,\ell)}, 1)$;

and both diagrams are drawn in such a way as to create no double-crossings. The degree of A^σ_{s} can be calculated inductively as follows,

$$\deg(A^\sigma_{\mathsf{s}}) = \deg(\overline{A}^\sigma_{\mathsf{s}{\downarrow}_{\{1,\dots,n-1\}}}) + \deg(1^{\lambda+(n,1,\ell)}_{\lambda+(r,c,m)}) \quad \text{where} \quad \deg(\overline{A}^\sigma_{\mathsf{s}{\downarrow}_{\{1,\dots,n-1\}}}) = \deg(A^\sigma_{\mathsf{s}{\downarrow}_{\{1,\dots,n-1\}}})$$

by construction. The degree of $1^{\lambda+(n,1,\ell)}_{\lambda+(r,c,m)}$ is calculated in terms of the number of crossings as in Definition 4.2. We calculate this brick-by-brick and diagonal-at-a-time as follows.

If S_n passes through a brick $\mathbf{B}_k$ for $k = 1$, $k = 2, 3$, $k = 4, 5$ or $k = 6$, then the degree contribution of this crossing is 0, -1, $+1$, or -2 respectively. Let $\mathbf{D}$ be a diagonal in the diagram 1_λ and suppose that S_n passes through $\mathbf{D}$. An addable diagonal is built out of a single $\mathbf{B}_k$ brick for $k \in \{4, 5, 6\}$ and some number (possibly zero) of $\mathbf{B}_1$ bricks. A removable (respectively invisible) diagonal has an extra single $\mathbf{B}_k$ brick for $k = 6$ (respectively $k \in \{2, 3\}$). Summing over the degrees, we conclude that the crossing of S_n with an addable, removable, or invisible i-diagonal has degree $+1$, -1, or 0. Finally, we observe that the i-diagonals in 1_λ which the S_n strand crosses are precisely those to the left of $\mathbf{I}^\sigma_{(r,c,m)}$ and so the total degree contribution of this strand is $|\mathcal{A}_{\mathsf{t}}(n)| - |\mathcal{R}_{\mathsf{t}}(n)|$. Therefore condition (1) holds. □

Using the standard facts about cellular algebras which we recalled in Section 2, we are now able to define the many different families of cell/Specht modules and many different parameterisations/constructions of irreducible modules promised in the introduction.

Definition 7.2. Fix an e-charge $\underline{s} = (e; s_0, s_1, \dots, s_{\ell-1}) \in \mathbb{N}_{>1} \times (\mathbb{Z}/e\mathbb{Z})^\ell$. For each integral lift $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ of the e-charge and $\lambda \in \mathscr{P}_n^\ell$, we let

$$\mathrm{S}^{\Bbbk}_\sigma(\lambda) = \{A^\sigma_{\mathsf{s}} \mid \mathsf{s} \in \mathrm{Std}_\sigma(\lambda)\} \tag{7.1}$$

denote the corresponding $\mathcal{H}_n^{\Bbbk}(\underline{s})$ cell-module. For $\Bbbk$ a field and an integral lift $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ of the e-charge, we set

$$\Sigma_n^\ell = \{\lambda \in \mathscr{P}_n^\ell \mid \mathrm{rad}(\langle\ ,\ \rangle_\lambda) \neq \mathrm{S}_\sigma^{\Bbbk}(\lambda)\} \subseteq \mathscr{P}_n^\ell$$

and we let

$$\{\mathrm{D}_\sigma^{\Bbbk}(\lambda) := \mathrm{S}_\sigma^{\Bbbk}(\lambda)/\mathrm{rad}^{\Bbbk}(\langle\ ,\ \rangle_\lambda) \mid \lambda \in \Sigma_n^\ell\}.$$

Proposition 7.3. *For $\sigma_0 \gg \sigma_1 \gg \cdots \gg \sigma_{\ell-1}$ an asymptotic charge, the basis*

$$\{A_{\mathsf{st}}^\sigma \mid \lambda \in \mathscr{P}_n^\ell, \mathsf{s}, \mathsf{t} \in \mathrm{Std}_\sigma(\lambda)\}$$

differs from that of [HM10, Main Theorem] *by $\rhd_\sigma$-unitriangular change of basis matrix. For each $\mu \in \mathscr{P}_n^\ell$ the cell module $\mathrm{S}_\sigma(\mu)$ is isomorphic as a graded $\mathcal{H}_n^{\Bbbk}(\underline{s})$-module to the usual graded Specht module (with the same label) defined in* [BKW11, Theorem 4.10].

Proof. For $\mathsf{t}^\lambda \in \mathrm{Std}_\sigma(\lambda)$ the *unique* tableau satisfying $\mathsf{t}^\lambda \trianglerighteq \mathsf{s}$ for all $\mathsf{s} \in \mathrm{Std}_\sigma(\lambda)$. The chain of 2-sided cell ideals in [HM10] is given by the $\mathcal{H}_n^{\Bbbk}(\underline{s}) y_\lambda \mathcal{H}_n^{\Bbbk}(\underline{s})$ for

$$y_\lambda = \prod_{k=1}^{n} y_k^{|\mathcal{A}_{\mathsf{t}^\lambda}(k)|} e_{\mathrm{res}(\mathsf{t}^\lambda)} \quad \text{and we claim that} \quad y_\lambda = A_{\mathsf{t}^\lambda \mathsf{t}^\lambda}^\sigma + \sum_{\substack{\mu \rhd_\sigma \lambda \\ \mathsf{s},\mathsf{t} \in \mathrm{Std}_\sigma(\mu)}} \alpha_{\mathsf{uv}} A_{\mathsf{st}}^\sigma$$

for $\alpha_{\mathsf{uv}} \in \mathbb{Z}$. Once the claim is established, we have that the chains of two-sided ideals are isomorphic and the result follows. We now prove the claim by induction on $\rhd_\sigma$ and $1 \leqslant k \leqslant n$. At the kth step, we pull the kth strand to the right using the leftmost relation in (A9) until we encounter a solid j-strand (from some earlier step in the process) with $|j - i_k| \leqslant 1$. If the kth strand is undecorated (either because $|\mathcal{A}_{\mathsf{t}^\lambda}(k)| = 0$ or because we have already applied (A9) a total of $|\mathcal{A}_{\mathsf{t}^\lambda}(k)|$ times) then the kth strand is now part of either an M_2-brick or has comes to rest in the required position (in which case we are done). In the former case, we use the $\mathbf{M}_2$ analogue of equation (5.2) to move the strand rightwards at the expense of an error term (which is zero by Proposition 6.7). Otherwise, we have that the kth strand carries a *single* dot (by definition of t^λ) and we use (A6) to move the strand rightwards at the expense of an error term (which is again zero by Proposition 6.7). Repeating as necessary, the process terminates when the kth strand reaches the x-coordinate of the box $(\mathsf{t}^\lambda)^{-1}(k)$. □

We now consider $\mathcal{H}_2^{\Bbbk}(\underline{s})$ for $\underline{s} = (e; s_0, s_1) = (2; 0, 1)$. This algebra has two irreducible modules, $\mathrm{D}^{\Bbbk}(0,1)$ and $\mathrm{D}^{\Bbbk}(1,0)$, which are generated by $e(0,1)$ and $e(1,0)$ respectively, and which are annihilated by all the other generators of $\mathcal{H}_2^{\Bbbk}(\underline{s})$. Here we label the irreducibles by the corresponding idempotents (not by ℓ-partitions) because this labelling is independent of the charge; we will reconcile this with the charged labelings in Examples 10.6 and 10.7. There are two charges, $\sigma = (e; \sigma_0, \sigma_1) = (2; 4, 1)$ and $\sigma = (e; \sigma_0', \sigma_1') = (2; 2, 1)$, which give rise to distinct cellular structures (every other charge gives a basis equivalent to one of these). We show that there is no isomorphism relating the sets of cell modules obtained from these distinct charges.

Example 7.4. Let $\sigma = (e; \sigma_0, \sigma_1) = (2; 4, 1)$, we remark that this is a well-separated charge. The σ-dominance order is as follows,

$$((2), \varnothing) \rhd_\sigma ((1^2), \varnothing) \rhd_\sigma ((1), (1)) \rhd_\sigma (\varnothing, (2)) \rhd_\sigma (\varnothing, (1^2)).$$

We let

$$\mathsf{w} \in \mathrm{Std}_\sigma((2), \varnothing)) \quad \mathsf{v} \in \mathrm{Std}_\sigma((1^2), \varnothing) \quad \mathsf{t}, \mathsf{u} \in \mathrm{Std}_\sigma((1), (1)) \quad \mathsf{s} \in \mathrm{Std}_\sigma(\varnothing, (2)) \quad \mathsf{r} \in \mathrm{Std}_\sigma(\varnothing, (1^2)).$$

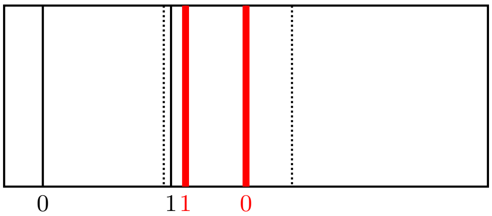
We choose t so that the box $\mathsf{t}^{-1}(1)$ has residue 0. We leave constructing the diagrammatic version of this basis as an exercise for the reader. Instead, we describe the basis as a linear combination of products of the KLR generators (using the process described in Theorem 6.17) as follows,

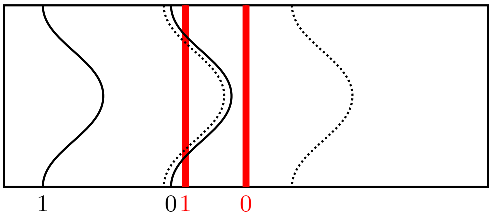
A^σ_{rr}	$= e(1,0)$		
A^σ_{ss}	$= y_2 e(1,0)$		
A^σ_{tt}	$= e(0,1)$	A^σ_{tu}	$= \psi_1 e(1,0)$
A^σ_{ut}	$= \psi_1 e(0,1)$	A^σ_{tt}	$= (y_2 - y_1)(y_1 - y_2)e(1,0)$
A^σ_{vv}	$= y_2 e(1,0)$		
A^σ_{ww}	$= y_2^2 e(1,0)$		

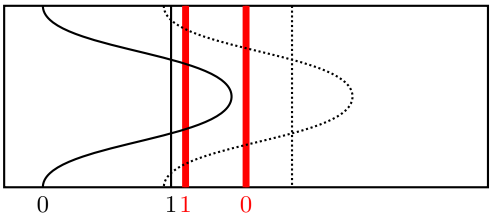
Example 7.5. Let $\sigma = (e; \sigma_0, \sigma_1) = (2; 2, 1)$. The $(2; 2, 1)$-dominance order is given as follows,

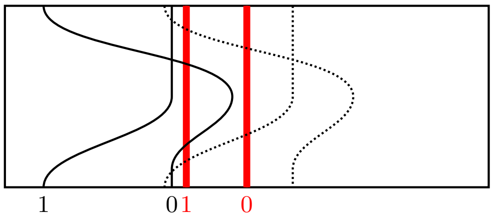
$$((1^2), \varnothing), (\varnothing, (1^2)) \lhd_\sigma ((1), (1)) \lhd_\sigma ((2), \varnothing), (\varnothing, (2)).$$

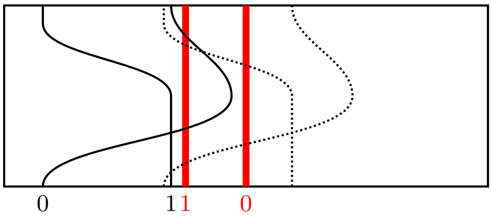
We let

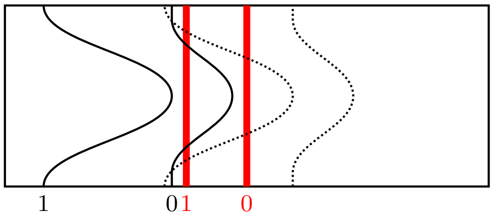
$A^\sigma_{\mathsf{rr}} =$ 

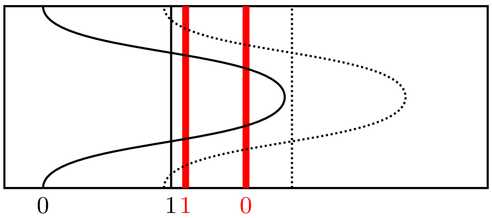
$A^\sigma_{\mathsf{ss}} =$ 

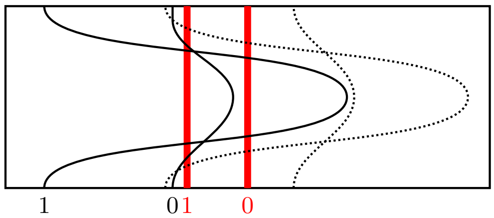
$A^\sigma_{\mathsf{tt}} =$ 

$A^\sigma_{\mathsf{tu}} =$ 

$A^\sigma_{\mathsf{ut}} =$ 

$A^\sigma_{\mathsf{uu}} =$ 

$A^\sigma_{\mathsf{vv}} =$ 

$A^\sigma_{\mathsf{ww}} =$ 

where we have chosen t so that the box $\mathsf{t}^{-1}(1)$ has residue 1. The elements $A^\sigma_{\mathsf{ww}}, A^\sigma_{\mathsf{vv}}$ are equal to the idempotents $e(1,0)$ and $e(0,1)$ respectively. Therefore these basis elements generate the corresponding irreducible modules $L^{\Bbbk}(1,0)$ and $L^{\Bbbk}(0,1)$ (modulo more dominant terms). One can rewrite the above basis elements using relation (A1) to (A13) (as in the proof of Theorem 6.17) to obtain a basis of this algebra in terms of a linear combination of products of the KLR-generators as follows,

A^σ_{rr}	$= e(1,0)$	A^σ_{ss}	$= e(0,1)$
A^σ_{tt}	$= y_2 e(1,0) - y_1 e(1,0)$	A^σ_{tu}	$= \psi_1 e(0,1)$
A^σ_{ut}	$= \psi_1 e(1,0)$	A^σ_{tt}	$= y_2 e(0,1) - y_1 e(0,1)$
A^σ_{vv}	$= y_2^2 e(1,0) - y_1 y_2 e(1,0)$	A^σ_{ww}	$= y_2^2 e(0,1) - y_1 y_2 e(0,1)$

We remark that any term with a y_1 in the product is zero by relation 3.13. These terms have been included in order to facilitate comparison with the diagrams.

Example 7.6. The graded dimension of $\mathcal{H}_2^{\Bbbk}(2;0,1)$ can be calculated using either the $(2;4,1)$ or $(2;2,1)$ cellular structure

$$(1)^2+(t)^2+(1+t^2)^2+(t)^2+(t^2)^2=2+4t^2+2t^4=(1)^2+(1)^2+(t+t)^2+(t^2)^2+(t^2)^2,$$

respectively and is (of course!) independent of the choice of cellular structure.

In [BKW11] the authors prove a series of results on asymptotic cellular structures: they construct graded tableaux-theoretic bases, analyse the restriction of asymptotic cell modules, and provide quasi-Garnir relations which serve as a warm-up to [KMR12]. We have already generalised the graded tableaux and resulting bases in Theorem 7.1 and we will prove the generalised restriction rule in Section 12. We now generalise their quasi-Garnir relations to all charges.

Theorem 7.7. *Given* t *a tableau of shape* λ*, we let* A_{t} *be a reduced diagram for* t*. We have that*

$$e(\imath)A^\sigma_{\mathsf{t}}=\delta_{\imath,\mathsf{res}(\mathsf{t})}A^\sigma_{\mathsf{t}}\quad y_rA^\sigma_{\mathsf{t}}=\sum_{\mathsf{s}\rhd_\sigma\mathsf{t}}\alpha_{\mathsf{s}}A^\sigma_{\mathsf{s}}\quad \psi_rA^\sigma_{\mathsf{t}}=\begin{cases}A^\sigma_{s_{r,r+1}(\mathsf{t})} & \text{if } \mathsf{t}\rhd_\sigma s_{r,r+1}(\mathsf{t})\in \mathrm{Std}_\sigma(\lambda)\\ \sum_{\mathsf{s}\rhd_\sigma s_{r,r+1}(\mathsf{t})}\beta_{\mathsf{s}}A^\sigma_{\mathsf{s}} & \text{otherwise.}\end{cases}$$

Proof. This result follows directly from Propositions 6.9 and 6.12 once we have shown that the Bruhat ordering on diagrams coincides with the dominance order on $\mathrm{Std}_\sigma(\lambda)$. Each solid (respectively ghost) strand in A^σ_{t} terminates at some northern point $\mathbf{I}^\sigma_{(p,1,\ell)}$ (respectively $\mathbf{I}^\sigma_{(p,1,\ell)}+1$) and the corresponding southern point $\mathbf{I}^\sigma_{\mathsf{t}^{-1}(p,1,\ell)}$ (respectively $\mathbf{I}^\sigma_{\mathsf{t}^{-1}(p,1,\ell)}+1$) for some associated integer $1\leqslant p\leqslant n$. A pair of solid or ghost strands in A_{t} associated to integers $1\leqslant p<q\leqslant n$ crosses if and only if $\mathbf{I}^\sigma_{(p,1,\ell)}<\mathbf{I}^\sigma_{(q,1,\ell)}$. Thus undoing a crossing of strands is equivalent to swapping the entries p and q in t to obtain a diagram A_{s} associated to $\mathsf{s}=s_{p,q}(\mathsf{t})$. Finally, we observe that $\mathbf{I}^\sigma_{(p,1,\ell)}<\mathbf{I}^\sigma_{(q,1,\ell)}$ implies $\mathsf{s}\lhd_\sigma\mathsf{t}$, as required. □

8. Generic semisimplicity and the decomposition map over $\mathbb{Q}$

We now recall Webster's definition of a generically semisimple algebra which specialises to be isomorphic to the (graded) quiver Cherednik and Hecke algebras of this paper (over $\mathbb{Q}$ or $\mathbb{C}$). This generic semisimplicity allows us to understand the many "charged" families of Specht modules as specialisations of a single family of semisimple modules (via many *different* integral forms on these modules). In more detail, we now recall Webster's definition of an algebra $\mathbb{B}_n^{\Bbbk}(\sigma,(q;Q_0,Q_1,\dots,Q_{\ell-1}))$ associated to $n\in\mathbb{N}$ and parameters q and $Q_0,Q_1,\dots,Q_{\ell-1}$. When we specialise $q=\xi$ to a primitive eth root of unity and $Q_m=q^{\sigma_m}$ for $0\leqslant m<\ell$ we will see that $\mathbb{A}_n^{\mathbb{Q}}(\sigma)\cong\mathbb{B}_n^{\mathbb{Q}}(\sigma,(\xi;\xi^{\sigma_0},\xi^{\sigma_1},\dots,\xi^{\sigma_{\ell-1}}))$.

8.1. **Algebra definition.** We now define the algebra of interest. Our definition is slightly reverse-engineered in order to make it easier to understand the isomorphism (see Remark 8.5). We first require a "generic" versions of definitions of the residues and contents from Section 1. We assume all the notation and definitions of Section 1. We define the (q,Q)-content of a box as follows,

$$\mathsf{ct}_{q,Q}(r,c,m)=q^{c-r}Q_m$$

and we set $\mathsf{ct}_{q,Q}(\lambda)=\sum_{(r,c,m)\in\lambda}\mathsf{ct}_{q,Q}(r,c,m)$. Upon specialisation of $q=\xi$ and $Q_m=\xi^{\sigma_m}$ for $0\leqslant m<\ell$, we have

$$\mathsf{ct}_{q,Q}(r,c,m)|_{q=\xi,Q_m=\xi^m}:=\xi^{c-r}\xi^{\sigma_m}=\xi^{\mathsf{res}(r,c,m)}.$$

Definition 8.1. Given a σ-diagram of type (μ,λ), we define a corresponding degraded σ-diagram to be any diagram obtained by relabelling (and recolouring) as follows. We recolour each solid strand as a green double-edged line and replace the residue of this solid strand with some (q,Q)-content q^iQ_m for $i\in\mathbb{Z}$ and $0\leqslant m<\ell$; we relabel the residue of the σ_m red strand with Q_m.

Definition 8.2. The associative $\Bbbk$-algebra, $\mathbb{B}_n^{\Bbbk}(\sigma,(q;Q_0,Q_1,\dots,Q_{\ell-1}))$, is generated (as a $\Bbbk$-module) by all inequivalent degraded σ-diagrams modulo the local relations (B1) to (B12) below. The product b_1b_2 of two diagrams $b_1,b_2\in\mathbb{B}_n^{\Bbbk}(\sigma,(q;Q_0,Q_1,\dots,Q_{\ell-1}))$ is then given by putting b_1 on top of b_2. This product is defined to be 0 unless the southern border of b_1 is given by the same loading as the northern border of b_2 with $(q;Q)$-contents matching in the obvious manner, in which case we obtain a new diagram with loading inherited from those of b_1 and b_2.

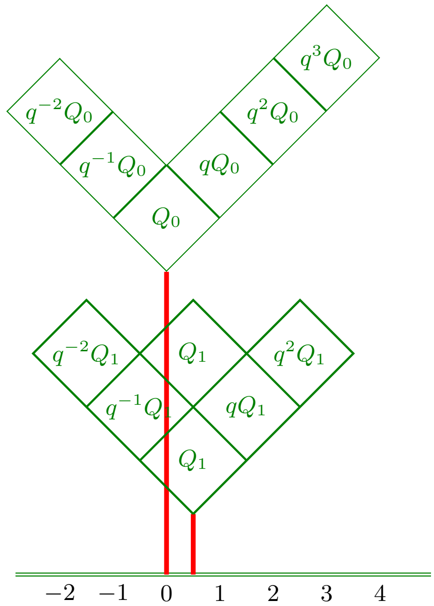


FIGURE 20. We picture the (q, Q)-contents of the 2-partition $((4, 1^2) \mid (3, 2, 1))$ for $\sigma = (0, 1)$ and $(0, 4)$ respectively. In each box we have placed the (q, Q)-content of the box.

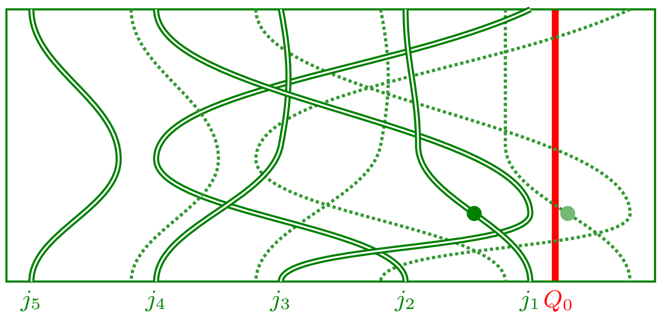


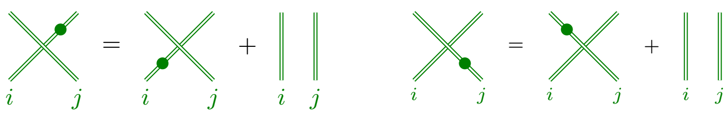
FIGURE 21. A σ-diagram, $B \in \mathbb{B}_5^{\Bbbk}(0, (q; Q_0))$, with northern and southern loading $\mathbf{I}^\sigma_\omega$ for $\omega = (1^5)$. Here $j_k = q^{i_k} Q_0$ for some $i_k \in \mathbb{Z}$ and $1 \leqslant k \leqslant 5$.

(B1) Any diagram may be deformed isotopically; that is, by a continuous deformation of the diagram which avoids tangencies, double points and dots on crossings.

(B2) Any solid dot can pass through an arbitrary crossing involving a ghost strand. Namely:

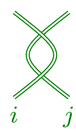
for i, j any (q, Q)-contents and their images through reflection in the vertical axis hold.

(B3) We can pass a solid dot through a crossing at the expense of an error term:



for i, j any (q, Q)-contents. Ghost dots can pass through any crossing of strands freely.

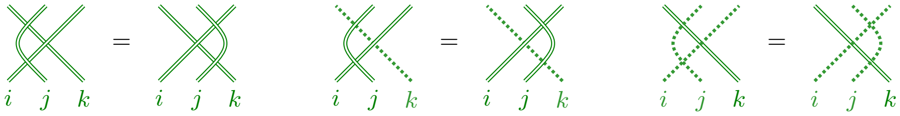
(B4) For i, j any (q, Q)-contents, a double-crossings of solid strands is zero, that is

$0 =$ 

(B5) For i, j any (q, Q)-contents, a double-crossing of ghost and solid strands can be undone as follows:

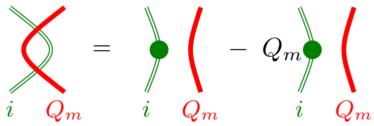
(B6) We can pull a solid strand through a ghost-crossing (or a ghost strand through a solid-crossing) at the expense of an error term: for i, j, k any (q, Q)-contents we have

(B7) All other triples of solid and ghost strands satisfy the naive braid relation:



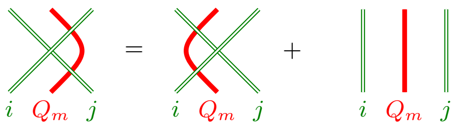
for any i, j, k any (q, Q)-contents and their mirror images through reflection in the vertical axis hold. Performing the leftmost relation implicitly involves manipulating a braid of three ghost strands at the same time (which we do not picture) and vice versa.

(B8) Double-crossings of solid and red strands can be undone at the expense of an error term



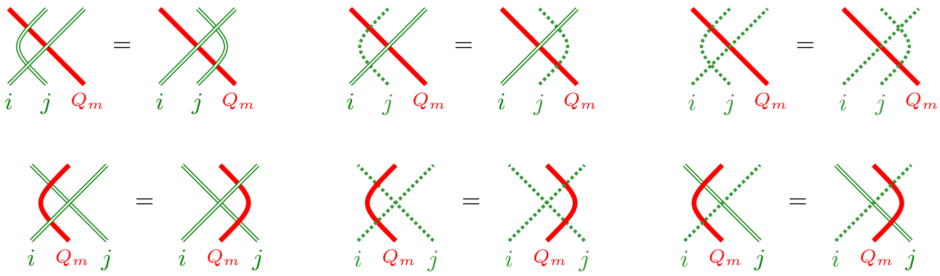
for $0 \leqslant m < \ell$ and i any (q, Q)-content; the mirror image through reflection through the vertical axis also holds. Ghost strands and ghost dots may pass through red strands freely.

(B9) Solid crossings and dots can pass through red strands, with a correction term,



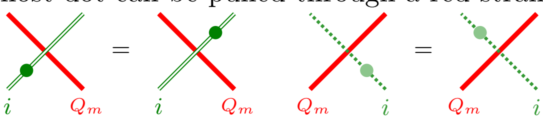
for $0 \leqslant m < \ell$ and i, j any (q, Q)-contents.

(B10) Any braid involving a red strand and not of the form in (B9) can be undone without cost:



for $0 \leqslant m < \ell$ and i, j any (q, Q)-contents; their reflections through the vertical axis hold.

(B11) Finally, any solid or ghost dot can be pulled through a red strand without cost:



for $0 \leqslant m < \ell$ and i any (q, Q)-content; their reflections through the vertical axis also hold.

We note that the diagram ϵ_λ whose solid points along the northern and southern boundaries are given by $\mathbf{I}_\lambda^\sigma$ for $\lambda \in \mathscr{C}_n^\ell$ and with *no crossing strands* is an idempotent by construction. We refer to any such diagram as a degraded weight idempotent. Finally, we have the following non-local idempotent relation.

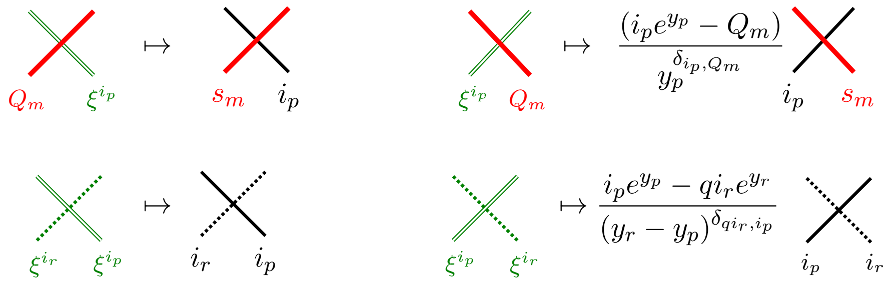
(B12) Any degraded weight idempotent, ϵ_λ, in which a solid strand is at least n units to the right of the rightmost red-strand is referred to as unsteady and set to be equal to zero.

Remark 8.3. Given (degraded) weight idempotents $\epsilon^{\imath}_\lambda$ and $1^{\imath}_\lambda$, we enumerate the solid (green or black, respectively) strands $1,\dots,n$ from right-to-left. We let $X_p\epsilon^{\imath}_\lambda$ and $y_p1^{\imath}_\lambda$ denote the diagrams obtained by adding a single dot on the pth solid strand (and a corresponding ghost dot on its ghost strand) and we let $e^{y_p}1^{\imath}_\lambda=\sum_{i\geqslant 0}\frac{1}{(i!)}y_p^i1^{\imath}_\lambda$. We set $\epsilon^{\sigma}_\omega=\sum_{\imath}\epsilon^{\imath}_\omega$ where the sum is over all (q,Q)-content sequences.

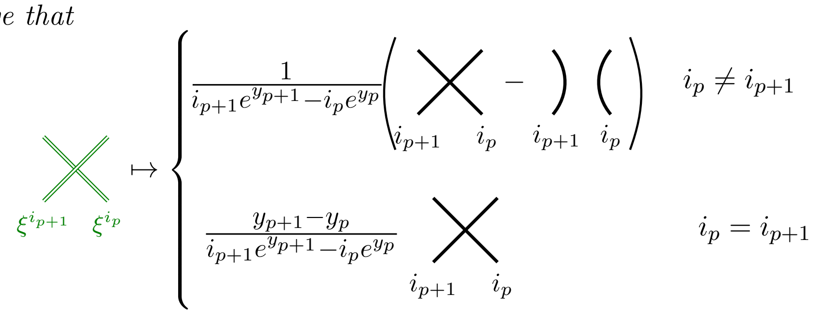
Theorem 8.4 ([Web17, Theorem 4.6] and [Web20, Theorem 6.9])**.** *We have an isomorphism of* $\mathbb{Q}$*-algebras* $\zeta:\mathbb{B}^{\mathbb{Q}}_n(\sigma(\xi;\xi^{\sigma_0},\xi^{\sigma_1},\dots,\xi^{\sigma_{\ell-1}}))\to\mathbb{A}^{\mathbb{Q}}_n(\sigma)$*. This isomorphism is given by specifying what happens on every local region of a diagram, as follows. We have that*

$$\epsilon^{\imath}_\lambda\mapsto 1^{\imath}_\lambda \qquad X_p\epsilon^{\imath}_\lambda\mapsto e^{y_p}1^{\imath}_\lambda$$

together with the flips of the latter two diagrams through the horizontal axis. We have that



and finally, we have that



Remark 8.5. Webster takes a slightly different approach the definition of this algebra. He defines the algebra as above, but does not attach $(q;Q)$-contents to the strands. Webster then observes the following. Let M be a finite dimensional $\mathbb{B}^{\Bbbk}_n(\sigma,(q;Q_0,Q_1,\dots,Q_{\ell-1}))$-module, the eigenvalues of each $X_p\epsilon_\lambda$ on M are of the form q^iQ_m for some $i\in\mathbb{Z}$ and $0\leqslant m<\ell$. So M decomposes as a direct sum of its weight spaces

$$M_{\underline{i}}=\{v\in M\mid (X_p-q^{i_p}Q^{m_p})^Nv=0\text{ for all }p=1,\dots,n\text{ and }N\gg 0\}$$

Considering the weight-space decomposition of the regular module, one deduces that there is a system $\{\epsilon^{\imath}_\lambda\mid \imath_p=q^{i_p}Q_{m_p}\text{ for some }i_p\in\mathbb{Z}\text{ and }0\leqslant m_p<\ell, 1\leqslant p\leqslant n\}$. With this notation in place, Webster then decomposes the identity of his algebra as a sum of these idempotents — thus obtaining our $(q;Q)$-content decorated diagrams. In what follows, we will consider the summation over all possible the (q,Q)-content decorations on any given solid green strand; we denote the resulting diagram without decorations on solid strands.

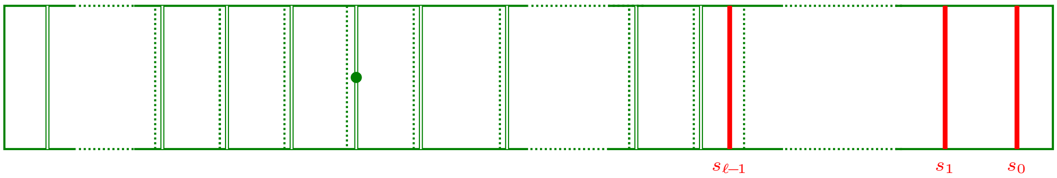
Remark 8.6. The following proposition and theorem are due to Webster and we provide citations here. However, we remark that, with only a few minor modifications, one can repeat all the arguments of Section 6 almost verbatim (one simply has to "forget the residues" of solid and ghost strands on account of relations (B1) to (B12) being residue-free). Indeed, all the results and proofs of Section 6 were very heavily based on ideas from [Web17] and [BKW11].

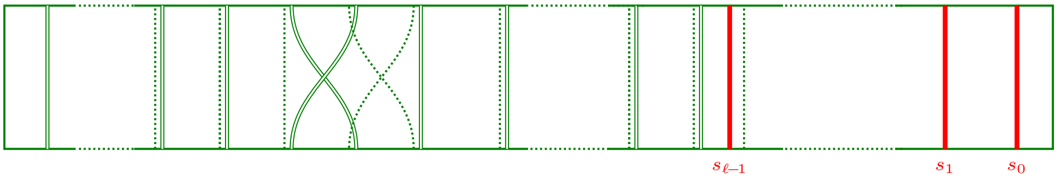
There is only one significant change to the analogues of the results from Section 6. Namely, we must replace the statement $y_{(r,c,m)}1_\mu\in\mathbb{A}^{\rhd\mu}_n(\sigma)$ in Proposition 6.7 with $X_{(r,c,m)}\epsilon_\mu\in\mathsf{ct}_{q,Q}(r,c,m)\epsilon_\mu+$

$\mathbb{B}_n^{\rhd\mu}(\sigma,((q;Q_0,Q_1,\dots,Q_{\ell-1})))$. To see this, one should compare the extra scalar on the righthand-side of relation (B8) versus its analogue in relation (A9) and the scalar on the righthand-side of relation (B5) versus its analogue, the pair of relations (A5), (A6). We revisit this idea in Proposition 8.10, below.

Proposition 8.7 ([Web20, Proposition 5.7])**.** *Let $\Bbbk$ be a integral domain. We have an isomorphism of $\Bbbk$-algebras*

$$\sigma_{q,Q}:H_n^{\Bbbk}(q;Q_0,Q_1,\dots,Q_{\ell-1})\to \mathsf{e}_\omega^\sigma\mathbb{B}_n^{\Bbbk}(\sigma,(q;Q_0,Q_1,\dots,Q_{\ell-1}))\mathsf{e}_\omega^\sigma.$$

$\sigma_{q,Q}(X_k)=$ 

$\sigma_{q,Q}(T_k+1)=$ 

where the dot is on the kth solid strand (from the right) and the crossing strands are the kth and $(k+1)$st solid strands from the right.

We let B_{ST} denote the degraded σ-diagram obtained from A_{ST} by relabelling each s_m-strand with the content Q_m and forgetting the residues of all other strands. In other words, we sum over all possible (q,Q)-contents on each green strand in B_{ST}.

Theorem 8.8. [Web17, Theorem 2.24] *Let $\Bbbk$ be an integral domain. The $\Bbbk$-algebra $\mathbb{B}_n^{\Bbbk}(\sigma,(q;Q_0,Q_1,\dots,Q_{\ell-1}))$ is free as an $\Bbbk$-module and has a cellular basis*

$$\{B_{\mathsf{ST}}\mid \mathsf{S}\in\mathrm{SStd}_\sigma(\lambda,\mu),\mathsf{T}\in\mathrm{SStd}_\sigma(\lambda,\nu),\lambda\in\mathscr{P}_n^\ell,\mu,\nu\in\mathscr{C}_n^\ell\}$$

with respect to the σ-dominance order on $\mathscr{P}_n^\ell$ and the involution $*$ given by horizontal reflection. We let $\Delta^{\Bbbk}_{\sigma,q,Q}(\lambda)$ denote the corresponding cell-module for $\lambda\in\mathscr{P}_n^\ell$.

Corollary 8.9. *Specialise $q=\xi$ and $Q_m=\xi^m$ for $0\leqslant m<\ell$ and $\Bbbk=\mathbb{Q}$. The $\mathcal{H}_n^{\mathbb{Q}}(\underline{s})$-modules $\mathrm{S}_\sigma^{\mathbb{Q}}(\lambda)$ and $\zeta(\mathsf{e}_\omega\Delta^{\mathbb{Q}}_{\sigma,q,Q}(\lambda))$ are isomorphic.*

Proof. We use the notation from the proof of Theorem 6.23 and we let $\mathsf{S},\mathsf{T}\in\mathrm{SStd}_\sigma(\lambda,\omega)$. We let $\widehat{A}_{\mathsf{S'T'}}\in\mathcal{H}_n^{\mathbb{Q}}(\underline{s})$ denote some diagram obtained from the graded cellular basis element $A_{\mathsf{S'T'}}\in\mathcal{H}_n^{\mathbb{Q}}(\underline{s})$ by adding some number (possibly zero) of dots along the strands. By the definition of the map ζ, we have that

$$\zeta(B_{\mathsf{ST}})=A_{\mathsf{ST}}+\sum_{\substack{\mathsf{U'},\mathsf{T'}\in\mathrm{SStd}_\sigma(\lambda,\omega)\\ \sharp(\mathsf{S'},\mathsf{T'})<\sharp(\mathsf{S},\mathsf{T})}}\widehat{A}_{\mathsf{S'T'}}.$$

Therefore by Propositions 6.7 and 6.9, we have that

$$\zeta(B_{\mathsf{ST}})\in A_{\mathsf{ST}}+\sum_{\substack{\mathsf{U'},\mathsf{T'}\in\mathrm{SStd}_\sigma(\lambda,\omega)\\ \sharp(\mathsf{S'},\mathsf{T'})<\sharp(\mathsf{S},\mathsf{T})}}k_{\mathsf{S'T'}}A_{\mathsf{S'T'}}+\mathsf{E}_\omega\mathbb{A}_n^{\rhd\lambda}(\sigma)\mathsf{E}_\omega$$

for some $k_{\mathsf{S'T'}}\in\Bbbk$. Thus the bases $\{\zeta(B_{\mathsf{ST}})\mid\mathsf{S},\mathsf{T}\in\mathrm{SStd}_\sigma(\lambda,\omega),\lambda\in\mathscr{P}_n^\ell\}$ and $\{A_{\mathsf{ST}}\mid\mathsf{S},\mathsf{T}\in\mathrm{SStd}_\sigma(\lambda,\omega),\lambda\in\mathscr{P}_n^\ell\}$ differ by uni-triangular change of basis matrix and the result follows. $\square$

Proposition 8.10. *For each $\lambda\in\mathscr{P}_\ell(n)$, the element $X_\lambda=\sum_{(r,c,m)\in\lambda}X_{(r,c,m)}\mathsf{e}_\lambda$ is central within $\mathsf{e}_\lambda\mathbb{B}_n^{\Bbbk}(\sigma(q;Q_0,Q_1,\dots,Q_{\ell-1}))\mathsf{e}_\lambda$ and acts on $\mathsf{e}_\lambda\Delta^{\Bbbk}_{\sigma,q,Q}(\mu)$ as the scalar $\mathsf{ct}_{q,Q}(\mu)$.*

Proof. Centrality follows immediately from the relations (B1) to (B12) and therefore X_λ acts as a scalar on any module, it remains to calculate this scalar. We have that

$$X_{(r,c,m)}\mathsf{e}_\lambda = \begin{cases} qX_{(r-1,c,m)}\mathsf{e}_\lambda + \mathbb{B}^{\rhd\lambda} & \text{for } r>1 \\ q^{-1}X_{(r,c-1,m)}\mathsf{e}_\lambda + \mathbb{B}^{\rhd\lambda} & \text{for } c>1 \\ Q_m X_{(r,c,m)}\mathsf{e}_\lambda + \mathbb{B}^{\rhd\lambda} & \text{for } r=c=1 \end{cases}$$

by the first case of relation (B5), second case of relation (B5), and relation (B8) respectively. It follows that $X_\lambda \mathsf{e}_\lambda = \mathsf{ct}_{q,Q}(\lambda)\mathsf{e}_\lambda + \mathbb{B}^{\rhd\lambda}$. For a cellular basis element of $\mathsf{e}_\lambda \Delta^{\Bbbk}_{\sigma,q,Q}(\mu)$, the result follows by induction on the Bruhat order and the analogue of Proposition 6.9. $\square$

Bringing together Propositions 8.7 and 8.10 and Theorem 8.8 we immediately deduce the following:

Theorem 8.11. *For $\lambda \in \mathscr{P}^\ell_n$, the $H^{\mathbb{Q}}_n(q; Q_0, Q_1, \dots, Q_{\ell-1})$-module $\mathsf{e}_\omega \Delta^{\mathbb{Q}}_{\sigma,q,Q}(\lambda)$ is isomorphic to the irreducible module upon which the central element $X_1 + \dots + X_n$ acts as the scalar $\mathsf{ct}_{q,Q}(\lambda)$.*

9. The many different graded decomposition matrices

In this section let $\Bbbk$ be an arbitrary field. We now prove the first statement of Theorem B: namely that the decomposition matrices of Hecke algebras are uni-triangular with respect to any of Lusztig's $\mathbf{a}_\sigma$-orderings. By Corollary 8.9 and Theorem 8.11 the modules $\mathrm{S}^{\mathbb{Q}}_\sigma(\lambda)$ are obtained from the usual semisimple Specht modules after specialisation of $q = \xi$ and $Q_m = \xi^m$ for $0 \leqslant m < \ell$. Thus by Proposition 2.3, upon forgetting the grading, the cellular decomposition matrices coincide with the usual definition of a decomposition matrix coming from a modular system.

Theorem 9.1. *Given a fixed $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$, the graded decomposition matrix of $\mathcal{H}^{\Bbbk}_n(\underline{s})$ with respect to the σ-cellular structure appears as a submatrix of the decomposition matrix of $\mathbb{A}^{\Bbbk}_n(\sigma)$ as follows,*

$$\sum_{k\in\mathbb{Z}} [\mathrm{S}^{\Bbbk}_\sigma(\lambda) : \mathrm{D}^{\Bbbk}_\sigma(\mu)\langle k\rangle] = \sum_{k\in\mathbb{Z}} [\Delta^{\Bbbk}_\sigma(\lambda) : L^{\Bbbk}_\sigma(\mu)\langle k\rangle]$$

for $\lambda \in \mathscr{P}^\ell_n$, $\mu \in \Sigma^\ell_n$. Here $\Sigma^\ell_n \subseteq \mathscr{P}^\ell_n$ is the subset for which $\mathrm{D}^{\Bbbk}_\sigma(\mu) := \mathsf{E}^\sigma_\omega L^{\Bbbk}_\sigma(\mu) \neq 0$; the set $\{\mathrm{D}^{\Bbbk}_\sigma(\mu) \mid \mu \in \Sigma^\ell_n\}$ provides a complete set of non-isomorphic irreducible $\mathcal{H}^{\Bbbk}_n(\underline{s})$-modules. This matrix is uni-triangular with respect to the ordering $\rhd_\sigma$ on $\mathscr{P}^\ell_n$.

Proof. The unitriangularity result is immediate from Theorem 7.1 and standard results on cellular algebras recalled explicitly in Section 2. The equality is immediate from [Gre07, (6.6b)Lemma]. $\square$

In [Ari96, Web17, RSVV16, Los16] it is shown that the decomposition matrix of $\mathbb{A}^{\Bbbk}_n(\sigma)$ is given by the Kashiwara–Lusztig canonical basis for an irreducible highest weight $U(\widehat{\mathfrak{sl}}_e)$-module and the entries are given by certain Kazhdan–Lusztig polynomials; these can be computed using an algorithm due to Uglov [Ugl00].

One of the main advantages of our new $\mathbb{Z}$-lattices is that they allow us to define generalisations of James' adjustment matrices. The theory of adjustment matrices gives us a way of factorising representation theoretic questions into two steps: firstly *specialise the parameter* $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ and study the non-semisimple algebra $\mathcal{H}^{\mathbb{Q}}_n(\sigma)$; then *reduce modulo* p by studying $\mathcal{H}^{\Bbbk}_n(\sigma) = \mathcal{H}^{\mathbb{Z}}_n(\sigma) \otimes_{\mathbb{Z}} \Bbbk$. This allows us to factorise the problem of understanding decomposition matrices as follows,

$$[\mathrm{S}^{\Bbbk}_\sigma(\lambda) : \mathrm{D}^{\Bbbk}_\sigma(\mu)] = \sum_\nu [\mathrm{S}^{\mathbb{Q}}_\sigma(\lambda) : \mathrm{D}^{\mathbb{Q}}_\sigma(\nu)] \times [\mathrm{D}^{\mathbb{Q}}_\sigma(\nu) \otimes_{\mathbb{Z}} \Bbbk : \mathrm{D}^{\Bbbk}_\sigma(\mu)]. \tag{9.1}$$

On the right-hand side of the equality we have two matrices: the first is the σ-decomposition matrix for $\mathcal{H}^{\mathbb{Q}}_n(\underline{s})$ and the second is what we refer to as the generalised James' σ-adjustment matrix $\mathbf{Ad}^{\Bbbk}_\sigma(t)$. We emphasise that the definition of $\mathbf{Ad}^{\Bbbk}_\sigma(t)$ only makes sense because, by Theorem A, we have $\mathbb{Z}$-forms for the cell and irreducible modules which allow us to reduce modulo p in equation (9.1).

Example 9.2. The action of the generators on the basis of $\mathrm{S}^{\Bbbk}_{(2;2,1)}((1),(1))$ in Example 7.5 is given as follows,

$$\psi_1, y_1, y_2, e(0,0), e(1,1) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad e(0,1) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad e(1,0) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and therefore this module is a direct sum of the irreducible modules $L^{\Bbbk}(0,1)$ and $L^{\Bbbk}(1,0)$. Clearly the module $\mathrm{S}^{\Bbbk}_{(2;2,1)}((1),(1))$ is not cyclic. The action of the generators on the basis of $\mathrm{S}_{(2;4,1)}((1),(1))$ in Example 7.4 is given as follows,

$$y_1, y_2, e(0,0), e(1,1) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad e(0,1) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad e(1,0) \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \psi_1 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and therefore this module is a non-split extension of the irreducible modules $L^{\Bbbk}(0,1)$ and $L^{\Bbbk}(1,0)$. All the cell-modules for the charge $(2;2,1)$ are all indecomposable, whereas this is not the case for the charge $(2;4,1)$. Hence, there is no isomorphism relating the sets of cell modules from these two distinct charges. Notice that the two modules have the same composition factors, but not the same structure.

Example 9.3. The graded decomposition matrices with respect to these cellular bases are,

$$\mathbf{D}_{(2;2,1)}(t) = \begin{array}{r|cc} & \mathrm{D}^{\Bbbk}(1,0) & \mathrm{D}^{\Bbbk}(0,1) \\ \hline (\varnothing,(1^2)) & 1 & 0 \\ ((1^2),\varnothing) & 0 & 1 \\ ((1),(1)) & t & t \\ (\varnothing,(2)) & t^2 & 0 \\ ((2),\varnothing) & 0 & t^2 \end{array} \qquad \mathrm{D}_{(2;4,1)}(t) = \begin{array}{r|cc} & \mathrm{D}^{\Bbbk}(1,0) & \mathrm{D}^{\Bbbk}(0,1) \\ \hline (\varnothing,(1^2)) & 1 & 0 \\ (\varnothing,(2)) & t & 0 \\ ((1),(1)) & t^2 & 1 \\ ((1^2),\varnothing) & 0 & t \\ ((2),\varnothing) & 0 & t^2 \end{array}.$$

We cannot obtain $\mathbf{D}^{\Bbbk}_{(2;2,1)}(t)$ by permuting the rows of $\mathbf{D}^{\Bbbk}_{(2;4,1)}(t)$. However, letting $t=1$ we find that $\mathbf{D}^{\Bbbk}_{(2;2,1)}(1)$ can be obtained from by permuting the rows of $\mathbf{D}^{\Bbbk}_{(2;4,1)}(1)$.

10. Uglov combinatorics and the many different constructions of irreducible modules

In this section let $\Bbbk$ be an arbitrary field. In this section we complete the proof of Theorem B of the introduction. Namely, we provide many explicit constructions of the irreducible modules of (quiver) Hecke algebras (in terms of cellular bilinear forms) over arbitrary fields.

Definition 10.1. Fix $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^{\ell}$. Given $\lambda \in \mathscr{P}^{\ell}_n$ and $i \in \mathbb{Z}/e\mathbb{Z}$, we define the i-sequence of λ to be the sequence of addable and removable nodes (recorded by A and R respectively) in increasing order with respect to $\lhd_{\sigma}$. We define the reduced i-sequence to be the sequence of the form $R, R, \dots, R, A, A, \dots, A$ obtained from the above by repeatedly removing all pairs of the form (A, R). We say that the removable i-node of λ is σ-good if it corresponds to the rightmost R in the reduced i-sequence.

Definition 10.2. Given a fixed $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^{\ell}$, the set of Uglov ℓ-partitions $\Sigma^{\ell}_n \subseteq \mathscr{P}^{\ell}_n$ is defined recursively as follows. We have that $\varnothing \in \Sigma^{\ell}_n$. For $\lambda \in \mathscr{P}^{\ell}_n$, we have that $\lambda \in \Sigma^{\ell}_n$ if and only if there exists $i \in \mathbb{Z}/e\mathbb{Z}$ and a good i-node $\alpha \in \mathrm{Rem}_i(\lambda)$ such that $\lambda - \alpha \in \Sigma^{\ell}_n$.

Example 10.3. For asymptotic charges the Uglov ℓ-partitions defined above are better known as the *Kleshchev* ℓ-partitions.

We now recall [Jac07, Main Theorem], modifying the statement slightly by inputting the definition of a canonical basic set (Definition 2.1) and by having explicitly defined Σ^{ℓ}_n in Definition 10.2 using the "crystal combinatorics" made explicit in Definition 10.1.

Theorem 10.4 ([Jac07, Main Theorem]). *For each $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^{\ell}$, the algebra $H^{\mathbb{Q}}_n(\underline{s})$ has canonical basic set $\Sigma^{\ell}_n \subseteq \mathscr{P}^{\ell}_n$ with respect to the ordering $>_{\sigma}$.*

The following theorem extends Theorem 10.4 to arbitrary fields and also gives an explicit construction of the irreducible modules labelled by $\Sigma^{\ell}_n \subseteq \mathscr{P}^{\ell}_n$ (this is new even in the case of $\Bbbk = \mathbb{Q}$). This extends Ariki–Mathas's results for *asymptotic* charges [Ari01, AM00] to arbitrary charges.

Theorem 10.5. *Let $\Bbbk$ be an arbitrary field and let $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^{\ell}$. The irreducible $\mathcal{H}^{\Bbbk}_n(\underline{s})$-modules are constructed as follows*

$$\mathrm{D}^{\Bbbk}_{\sigma}(\lambda) := \mathrm{S}^{\Bbbk}_{\sigma}(\lambda)/\mathrm{rad}^{\Bbbk}(\langle\ ,\ \rangle_{\lambda})$$

are indexed by the set $\Sigma^{\ell}_n = \{$Uglov ℓ-partitions with respect to the pair $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^{\ell}\}$.

Proof. We first fix our field to be $\mathbb{Q}$. In Theorem 8.11 and Corollary 8.9 we proved that our cell-modules $\mathrm{S}^{\mathbb{Q}}_{\sigma}(\lambda)$ are obtained via specialisation from the irreducible modules of the semisimple Hecke algebra; moreover we showed that this preserved the labelling of these modules. We saw in Theorem 9.1 that the cellular structure of Theorem 7.1 gives rise to a unitriangular decomposition matrix (and hence a canonical basic set) with respect to to the ordering $\rhd_\sigma$ (which is a coarsening of $>_\sigma$) on $\mathscr{P}^\ell_n$. By Proposition 2.3 and Theorem 10.4 it follows that $\Sigma^\ell_n = \{$Uglov ℓ-partitions with respect to the pair $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell\}$ for $\Bbbk = \mathbb{Q}$. It remains to prove that the result extends to arbitrary fields, by reduction modulo p (with respect to the $\mathbb{Z}$-lattices of Theorem 7.1).

Let $\Bbbk$ be an arbitrary field. From the above, we know that $\mathrm{rad}^{\mathbb{Q}}(\langle\ ,\ \rangle_\lambda) = \mathrm{S}^{\mathbb{Q}}_\sigma(\lambda)$ for any $\lambda \notin \Sigma^\ell_n$. We also know that the number of irreducibles of the Hecke algebra is independent of the characteristic of the field [AM00] and that all irreducibles (regardless of the field) are obtained as quotients of these radicals by cellularity. Therefore, $\mathrm{rad}^{\Bbbk}(\langle\ ,\ \rangle_\lambda) = \mathrm{S}^{\Bbbk}_\sigma(\lambda)$ for any $\lambda \notin \Sigma^\ell_n$ by base change (as our bases are constructible over $\mathbb{Z}$). □

Example 10.6. The irreducible modules for $\sigma = (2; 2, 1)$ are labelled by $\{((1^2), \varnothing), (\varnothing, (1^2))\}$.

Example 10.7. The irreducible modules for $\sigma = (2; 4, 1)$ are labelled by $\{(\varnothing, (2)), (\varnothing, (1^2))\}$.

To summarise: the parameterisations of irreducible modules given by Theorem 10.5 are precisely those of Ariki's categorification theorem. Thus Theorem 7.1 provides the integral cellular bases "predicted" by Ariki's categorification theorem. Theorem 10.5 explicitly constructs these irreducible modules in terms of radicals of cellular bilinear forms for the first time.

11. The many different filtrations of projective modules

Our many cellular bases allow us to obtain many different filtrations on any fixed projective $\mathcal{H}^{\Bbbk}_n(\underline{s})$-module. The search for these different filtrations was initiated by Geck–Rouquier [GR01].

Theorem 11.1. *Fix $\Bbbk$ a field, $\underline{s} = (e; s_0, s_1, \dots, s_{\ell-1}) \in \mathbb{N}_{>1} \times (\mathbb{Z}/e\mathbb{Z})^\ell$, and let $\mathbf{P}_{\underline{s}}$ be a fixed projective indecomposable $\mathcal{H}^{\Bbbk}_n(\underline{s})$-module. For each and every integral lift, $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$, the projective module $\mathbf{P}_{\underline{s}}$ admits a filtration*

$$0 = M^\sigma_1 \subset M^\sigma_2 \subset \cdots \subset M^\sigma_z = \mathbf{P}_{\underline{s}}$$

such that for each $1 \leqslant r \leqslant z$, we have $M^\sigma_r / M^\sigma_{r-1}$ is isomorphic to some $\mathrm{S}^{\Bbbk}_\sigma(\mu^{(r)})$ for $\mu^{(r)} \in \mathscr{P}^\ell_n$ up to grading shift. We have that $\mu^{(r)} \lhd_\sigma \mu^{(r-1)}$ for $1 \leqslant r \leqslant z$. In particular, every projective module admits many different cell-filtrations (up to grading shift), one for each cellular structure in Theorem A, or equivalently, one for each quasi-hereditary cover $\mathbb{A}^{\Bbbk}_n(\sigma)$ of $\mathcal{H}^{\Bbbk}_n(\underline{s})$.

Proof. Fix an arbitrary integral lift $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$. For $\lambda \in \mathscr{P}^\ell_n$, let $\mathbf{P}_\sigma(\lambda)$ denote the corresponding projective $\mathbb{A}^{\Bbbk}_n(\sigma)$-module. Then $\mathbf{P}_\sigma(\lambda)$ admits a cell-filtration (with respect to the cellular structure of Theorem 6.14) by standard facts concerning quasi-hereditary algebras (and Corollary 6.24). Therefore $\mathsf{E}^\sigma_\omega \mathbf{P}_\sigma(\lambda)$ is an indecomposable $\mathcal{H}^{\Bbbk}_n(\underline{s})$-module with a filtration by $\mathrm{S}^{\Bbbk}_\sigma(\mu)$ such that $\mu \trianglerighteq_\sigma \lambda$. Given any integral lift of our e-charge, a full set of projective $H^{\Bbbk}_n(\underline{s})$-modules are given by $\{\mathsf{E}^\sigma_\omega \mathbf{P}_\sigma(\lambda) \mid \lambda \in \Sigma^\ell_n\}$ and so the result follows. □

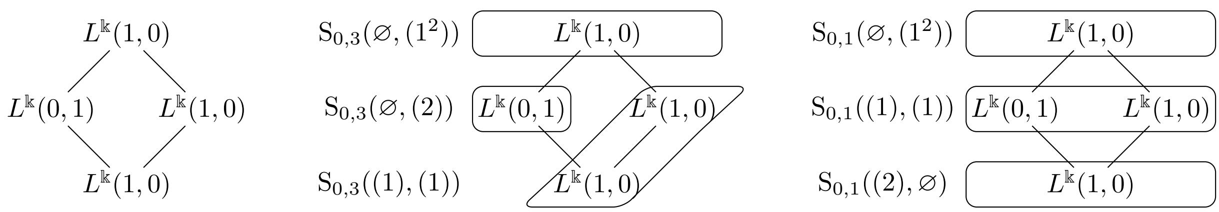


Figure 22. The projective cover of the irreducible module $L^{\Bbbk}(1,0)$ and its 2 distinct cell-filtrations. As both n and ℓ increase, we obtain many more distinct filtrations on each projective module.

Example 11.2. The algebra $\mathcal{H}_2^{\Bbbk}(2;0,1)$ has two indecomposable projective modules. We picture the full submodule structure of the projective $P^{\Bbbk}(1,0)$ in Figure 22. We also picture the two distinct cell-filtrations of this module for $\sigma=(2;4,1)$ and $\sigma=(2;2,1)$.

12. The restriction of a cell module for the quiver Hecke algebra

For every charge $\sigma\in\mathbb{N}_{>1}\times\mathbb{Z}^\ell$, we prove that the (graded) restriction of cell-module (down the tower of Hecke algebras) has a cell-filtration. We thus complete our program of generalising all the results from [BKW11] to arbitrary charges. This result is to be expected, given the 2-categorical origins of our $\mathbb{Z}$-bases [Web17] (where σ-diagrams arise in categorifying quantum knot variants). This result provides the key ingredient to the construction of resolutions of unitary modules for Cherednik algebras and algebraic varieties in [BNS].

Theorem 12.1. *Let $\Bbbk$ be a integral domain. Let $\lambda\in\mathscr{P}_n^\ell$ and let $\alpha_1\rhd_\sigma\alpha_2\rhd_\sigma\cdots\rhd_\sigma\alpha_z$ denote the removable boxes of λ, totally ordered according to the σ-dominance ordering. Then the restriction of a cell-module has an $\mathcal{H}_{n-1}^{\Bbbk}(\underline{s})$-module filtration*

$$0=\mathrm{S}_\sigma^{z+1,\lambda}\subset\mathrm{S}_\sigma^{z,\lambda}\subset\cdots\subset\mathrm{S}_\sigma^{1,\lambda}=\mathrm{Res}_{\mathcal{H}_{n-1}^{\Bbbk}(\underline{s})}(\mathrm{S}_\sigma^{\Bbbk}(\lambda))\tag{12.1}$$

such that, for each $1\leqslant r\leqslant z$, we have that

$$\mathrm{S}_\sigma^{\Bbbk}(\lambda-\alpha_r)\langle\deg(\alpha_r)\rangle\cong\mathrm{S}_\sigma^{r,\lambda}/\mathrm{S}_\sigma^{r+1,\lambda}.\tag{12.2}$$

Proof. For $1\leqslant r\leqslant z$, we define

$$\mathrm{S}_\sigma^{r,\lambda}=R\{A_{\mathsf{u}\mathsf{T}^\lambda}\mid\mathsf{u}\in\mathrm{Std}_\sigma(\lambda)\text{ and }\mathrm{Shape}(\mathsf{u}{\downarrow_{\{1,\dots,n-1\}}})\trianglerighteq\lambda-\alpha_r\}.$$

On the level of graded $\Bbbk$-modules, the chain of inclusions in equation (12.1) is clear. For $\mathsf{t}\in\mathrm{Std}_\sigma(\lambda-\alpha_r)$, we define $\varphi_r(\mathsf{t})\in\mathrm{Std}_\sigma(\lambda)$ to be the tableau obtained by adding the box α_r with entry n to the tableau $\mathsf{t}\in\mathrm{Std}_\sigma(\lambda-\alpha_r)$. Abusing notation, we define

$$\varphi_r(A_{\mathsf{t}\mathsf{T}^{(\lambda-\alpha_r)}})=A^\sigma_{\varphi_r(\mathsf{t})\mathsf{T}^\lambda}.$$

We assume that α_r is a box of residue $i\in\mathbb{Z}/e\mathbb{Z}$. It is clear that φ_r provides the required graded $\Bbbk$-module isomorphism of equation (12.2). It remains to verify that the chain of inclusions and the resulting isomorphisms hold on the level of $\mathcal{H}_{n-1}^{\Bbbk}(\underline{s})$-modules. We shall prove this by downward induction on the ordering on the removable nodes of λ. Let $\rho,\tau\in\mathscr{C}_n^\ell$ and suppose that $\rho\setminus(\rho\cap\tau)=\square_\rho:=(r_\rho,c_\rho,m_\rho)$ and $\tau\setminus(\rho\cap\tau)=\square_\rho:=(r_\tau,c_\tau,m_\tau)$. Let $\gamma\in\mathscr{C}_n^\ell$ and Given $\mathsf{S}\in\mathrm{SStd}_\sigma(\rho,\tau)$, $\mathsf{t}\in\mathrm{Std}_\sigma(\gamma)$ we define

$$\overline{\mathsf{S}}_\rho^\tau(r,c,m)=\begin{cases}\mathbf{I}^\sigma_{(r,c,m)} & \text{if }(r,c,m)\in\rho\cap\tau\\ \mathbf{I}^\sigma_{(r_\tau,c_\tau,m_\tau)} & \text{if }(r,c,m)=\square_\rho\end{cases}\qquad\bar{\mathsf{t}}(r,c,m)=\begin{cases}\mathsf{t}(r,c,m) & \text{if }(r,c,m)\in\nu\\ n & \text{if }(r,c,m)=\square_\rho\end{cases}$$

We have that

$$A_{\varphi_r(\mathsf{t})}=A_{\bar{\mathsf{t}}}\times A_{\mathsf{T}_\lambda^{\lambda-\alpha_r+(n,1,\ell)}}\tag{12.3}$$

for $\mathsf{t}\in\mathrm{Std}_\sigma(\lambda-\alpha_r)$. For $a\in\mathscr{H}_{n-1}^{\Bbbk}(\sigma)$ and $\mathsf{t}\in\mathrm{Std}_\sigma(\lambda-\alpha_r)$, it follows from Theorem 7.1 that

$$aA_{\mathsf{t}}=\sum_{\substack{\nu\trianglerighteq\lambda-\alpha_r\\ \mathsf{s}\in\mathrm{Std}_\sigma(\nu)\\ \mathsf{S}\in\mathrm{SStd}_\sigma(\nu,\lambda-\alpha_r)}}k_{\mathsf{s}\mathsf{S}}A_{\mathsf{s}\mathsf{S}}\tag{12.4}$$

for some $k_{\mathsf{s}\mathsf{S}}\in\Bbbk$. By equation (12.3), we have that

$$\varphi_r(aA_{\mathsf{t}})=\sum_{\substack{\nu\trianglerighteq\lambda-\alpha_r\\ \mathsf{s}\in\mathrm{Std}_\sigma(\nu)\\ \mathsf{S}\in\mathrm{SStd}_\sigma(\nu,\lambda-\alpha_r)}}k_{\mathsf{s}\mathsf{S}}A_{\bar{\mathsf{s}}\mathsf{S}}A_{\mathsf{T}_\lambda^{\nu+(n,1,\ell)}}\tag{12.5}$$

and so it will suffice to show that if $k_{\mathsf{s}\mathsf{S}}\neq0$ and $\nu\neq\lambda-\alpha_r$, then

$$A_{\bar{\mathsf{s}}}A^*_{\mathsf{S}}A_{\mathsf{T}^\lambda_{\nu+(n,1,\ell)}}\in\mathrm{S}_\sigma^{r+1,\lambda}\mod\mathbb{A}_n^{\rhd\lambda}(\sigma).\tag{12.6}$$

By Proposition 6.7, a necessary condition for $A^*_{\mathsf{S}} A_{\mathsf{T}_\lambda^{\nu+(n,1,\ell)}} \notin \mathbb{A}_n^{\rhd\lambda}(\sigma)$ is that $\lambda \trianglerighteq \nu + (n,1,\ell)$. On the other hand, we have that $\nu \rhd \lambda - \alpha_r$ by equation (12.4). Therefore, we need only consider terms in the sum 12.5 labelled by $\nu \in \mathscr{P}_n^\ell$ such that both

$$\nu \rhd \lambda - \alpha_r \text{ and } \lambda \trianglerighteq \nu + (n,1,\ell). \tag{12.7}$$

equation (12.7) implies that λ and ν only differ by moving some number (possibly zero) of boxes of residue $\mathsf{res}(\alpha_r) = i \in \mathbb{Z}/e\mathbb{Z}$. Again by equation (12.7), this implies that ν is obtained from $\lambda - \alpha_r$ by removing a non-zero (since $\nu = \lambda - \alpha_r$) set of i-boxes

$$\mathcal{R} = \{\alpha_{i_1} \rhd_\sigma \alpha_{i_2} \rhd_\sigma \cdots \rhd_\sigma \alpha_{i_S} \mid \alpha_r \rhd_\sigma \alpha_{i_s} \text{ for } 1 \leqslant s \leqslant S\} \subset \mathrm{Rem}_i(\lambda - \alpha_r) \tag{12.8}$$

and adding a set of i-boxes

$$\mathcal{A} = \{\alpha_{j_1} \rhd_\sigma \alpha_{j_2} \rhd_\sigma \cdots \rhd_\sigma \alpha_{j_S} \mid \alpha_r \trianglerighteq_\sigma \alpha_{j_s} \rhd_\sigma \alpha_{i_s} \text{ for } 1 \leqslant s \leqslant S\} \subseteq \mathrm{Add}_i(\lambda - \alpha_r) \tag{12.9}$$

such that $\mathcal{R} \neq \mathcal{A}$. We let N denote the set of all $\nu \in \mathscr{P}_{n-1}^\ell \setminus \{\lambda - \alpha_r\}$ which can be obtained from $\lambda - \alpha_r$ in this fashion. Putting all this together it will suffices to show that if $k_{\mathsf{sS}} \neq 0$ and $\mathsf{S} \in \mathrm{SStd}_\sigma(\nu, \lambda - \alpha_r)$ then in 12.5 and $\nu \in N$, then

$$A^*_{\mathsf{S}} A_{\mathsf{T}_\lambda^{\nu+(n,1,\ell)}} = A_{\mathsf{T}^\lambda_{\nu+(n,1,\ell)}} A_{\mathsf{S}} \in \mathbb{A}^{\rhd\lambda} \tag{12.10}$$

where we have applied the involution $*$ to simplify notation. By Proposition 6.11, we have that

$$A_{\mathsf{T}^\lambda_{\nu+(n,1,\ell)}} A_{\mathsf{S}} = A_{\mathsf{S}} A_{\mathsf{T}^{\nu+\alpha_r}_{\nu+(n,1,\ell)}} + \sum_{A''} A'' f_{A''}(y) \quad \text{for some } A' \rhd_\sigma A_{\mathsf{S}} A_{\mathsf{T}^{\nu+\alpha_r}_{\nu+(n,1,\ell)}} \text{ and } f_{A''}(y) \in \mathcal{Y}_{\nu+(n,1,\ell)}.$$

All terms on the righthand-side factor through the idempotent labelled by $\nu + \alpha_r$ which is strictly more dominant than λ by equation (12.7) and the result follows. $\square$

13. The generalised blob algebras and beyond

In the case of the symmetric groups modular representation theorists have long focussed on the subcategory of representations labelled by partitions with at most h columns for some $h \in \mathbb{Z}_{>0}$ over a field, $\Bbbk$, of characteristic (possibly much) greater than h. This subcategory is highest weight and far more amenable to study via the tools of Kazhdan–Lusztig theory [AJS94, RW] (in terms of the alcove geometry of type $A_{h-1} \subseteq \widehat{A}_{h-1}$). However, there is no obvious analogous subcategory/quotient algebra of $\mathcal{H}_n^{\Bbbk}(\underline{s})$ in higher levels; hence almost nothing is known or even conjectured about such Hecke algebras in positive characteristic. The purpose of this section is to introduce a candidate for such a quotient algebra and prove Martin–Woodcock's conjecture. The results of this section have been used by Libedinsky–Plaza as the basis of a modular analogue of Martin–Woodock's conjecture [LP]. Cox, Hazi and the author have subsequently proven this conjecture in [BCH] (using the ideas from this section).

13.1. **The cylindric charge.** Given $h \in \mathbb{N}$ we assume that our charge satisfies $h < \sigma_i - \sigma_j < e - h$ for $0 \leqslant i < j < \ell$. Let $\mathscr{P}_n^\ell(h) \subseteq \mathscr{P}_n^\ell$ denote the saturated subset consisting of all ℓ-partitions with at most h columns in any given component, that is,

$$\mathscr{P}_n^\ell(h) = \{\lambda = (\lambda^{(0)}, \ldots, \lambda^{(\ell-1)}) \mid \lambda_1^{(m)} \leqslant h \text{ for all } 0 \leqslant m < \ell\}.$$

For such a $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ our candidate quotient algebra is as follows:

$$\mathcal{Q}_{\ell,h,n}(\sigma) = \mathcal{H}_n^{\Bbbk}(\underline{s}) / \langle A^\sigma_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}_\sigma(\lambda), \lambda \notin \mathscr{P}_n^\ell(h)\rangle.$$

We will show in future work [BC18, BCH] that the category $\mathcal{Q}_{\ell,h,n}(\sigma)$-mod is incredibly rich and yet far more tractable than the category $\mathcal{H}_n^{\Bbbk}(\underline{s})$-mod: Under the restriction that $e > (h+1)\ell$, we shall cast representation theoretic questions in terms of an alcove geometry of type

$$A_{h-1} \times A_{h-1} \cdots \times A_{h-1} \subseteq \widehat{A}_{\ell h-1}.$$

In this section, we prove that $\mathcal{Q}_{\ell,h,n}(\sigma)$-mod is a highest-weight category over arbitrary field; thus generalises results on symmetric groups from [Erd97, Theorem 4.4] and results on the blob algebras of statistical mechanics [MS94, MW03]. We shall then prove Martin–Woodcock's conjecture.

13.2. **The representation theory of the algebras $\mathcal{Q}_{\ell,h,n}(\sigma)$.** With our definitions in place, we are now ready to prove that these algebras are quasi-hereditary and provide presentations of these algebras solely in terms of the classical KLR generators.

Theorem 13.1. *For $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ such that $h < \sigma_i - \sigma_j < e$ for $0 \leqslant i < j < \ell$, the algebra $\mathcal{Q}_{\ell,h,n}(\sigma)$ has a presentation solely in terms of the classical KLR generators as follows,*

$$\mathcal{Q}_{\ell,h,n}(\sigma) = \mathcal{H}_n^{\Bbbk}(\underline{s})/\langle e(\imath) \mid \imath \in (\mathbb{Z}/e\mathbb{Z})^{h+1} \text{ and } \imath_{k+1} = \imath_k + 1 \text{ for } 1 \leqslant k \leqslant h\rangle. \tag{13.1}$$

Over a field, the algebra $\mathcal{Q}_{\ell,h,n}(\sigma)$ is quasi-hereditary with irreducible modules indexed by $\mathscr{P}_n^\ell(h)$.

Proof. Consider an idempotent $e(\imath)$ of the form stated in equation (13.1). We pull the right most strand in $e(\imath)$ rightwards using the non-interacting relations. If $\imath_1 \neq \sigma_m$ for some $1 \leqslant m \leqslant \ell$ and we pull this strand rightwards until it is $> n$ units right of the red strand σ_0 and the diagram is zero by relation (A13). Otherwise $\imath_1 = s_m$ for some $1 \leqslant m \leqslant \ell$, and this process terminates when the solid σ_m-strand comes to rest upon reaching the vertical line with x-coordinate $(1,1,m)$. In other words, once it is ever-so-slightly to the right of the red s_m-strand with x-coordinate $\sigma_m - m/\ell$. By our assumptions on $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$, we can pull the solid (s_m+1)-strand rightwards until it reaches the vertical line with x-coordinate $(1,2,m)$. We then repeat this process until we have moved the right most $(h+1)$ solid strands as far right as possible. We let $\lambda = (\varnothing, \dots, \varnothing, (h+1), \varnothing, \dots, \varnothing, (0^{h+1}, 1^{n-1-h}))$ where the mth and ℓth are the only non-empty components of λ. The diagram produced by the process above is equal to A_{tt}^σ where $\varphi(\mathsf{t}) = \mathsf{T}$ is the tableau of shape λ and weight ω which takes $\mathsf{T}(r,1,m) = \mathbf{I}^\sigma_{(r,1,\ell)}$ for $1 \leqslant r \leqslant h+1$ and $\mathsf{T}(r,1,\ell) = \mathbf{I}^\sigma_{(r,1,\ell)}$ for $h+1 < r \leqslant n$; therefore $e(\imath) \in \{A_{\mathsf{st}}^\sigma \mid \mathsf{s},\mathsf{t} \in \mathrm{Std}_\sigma(\lambda), \lambda \notin \mathscr{P}_n^\ell(h)\}$.

Now for the reverse containment. By definition, there is no tableau $\mathsf{T} \in \mathrm{Std}_\sigma(\lambda)$ for $\lambda \in \mathscr{P}_n^\ell(h)$ with residue sequence $(s_m, s_m+1, \dots, s_m+h)$ and so $e(s_m, s_m+1, \dots, s_m+h)$ annihilates all these cell-modules. Hence the ideal by which we quotient in equation (13.1) has dimension less than or equal to $\sum_{\lambda \in \mathscr{P}_n^\ell \setminus \mathscr{P}_n^\ell(h)} |\mathrm{Std}_\sigma(\lambda)|^2$ and the reverse containment holds. Finally, the λth cell layer contains an idempotent e_{T_λ} for $\mathscr{P}_n^\ell(h)$ and so the algebra is quasi-hereditary, as required. □

Corollary 13.2. *The algebra $\mathcal{Q}_{\ell,1,n}(\sigma)$ is isomorphic to the generalised blob algebra of* [MW03].

Proof. We have seen that $\mathcal{Q}_{\ell,1,n}(\sigma)$ is the quotient of $\mathcal{H}_n^{\Bbbk}(\underline{s})$ by the two-sided ideal generated by $\sum_{0 \leqslant m < \ell} e(s_m, s_m+1)$. Each idempotent $e(s_m, s_m+1)$ in this sum spans a 1-dimensional irreducible $\mathcal{H}_2^{\Bbbk}(\underline{s})$-module labelled by the ℓ-partition $(\varnothing, \dots, \varnothing, (2), \varnothing, \dots, \varnothing)$. The result follows. □

Corollary 13.3. *The algebra $\mathcal{Q}_{1,h,n}(\sigma)$ is isomorphic to the generalised Temperley–Lieb algebra of* [Här99, Erd97] *and is Morita equivalent to the Ringel dual of the q-Schur algebra of* GL_h *(acting on n-fold q-tensor space).*

Proof. We have seen that $\mathcal{Q}_{1,h,n}(\sigma)$ is the quotient of $\mathcal{H}_n^{\Bbbk}(\underline{s})$ by the two-sided ideal generated by $e(s_m, s_m+1, \dots, s_m+h)$ which labels the trivial representation of $\mathcal{H}_{h+1}^{\Bbbk}(e;0)$. The result follows by [Här99, Theorem 4]. □

Theorem 13.4 (Martin Woodcock's conjecture [MW03])**.** *Let $\sigma \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ be such that $1 < s_i - s_j < e$ for $0 \leqslant i < j \leqslant \ell$. The graded decomposition matrix of $\mathcal{Q}_{\ell,1,n}(\sigma)$ appears as a square submatrix of that of $\mathcal{H}_n^{\Bbbk}(\underline{s})$ with respect to the σ-cell structure. We have that*

$$\sum_{k \in \mathbb{Z}} [\mathrm{S}_\sigma^{\mathbb{Q}}(\lambda) : \mathrm{D}_\sigma^{\mathbb{Q}}(\mu)\langle k\rangle] = n_{\lambda\mu}(t)$$

for $\lambda, \mu \in \mathscr{P}_n^\ell(1)$ where $n_{\lambda\mu}(t)$ is equal to a non-parabolic affine Kazhdan–Lusztig polynomial of type $\widehat{A}_{\ell-1}$. The action of the affine Weyl group on $\mathscr{P}_n^\ell(1)$ is given as in [BCS17, Section 3].

Proof. The square shape of the decomposition matrix follows from quasi-heredity Theorem 13.1. The algebra $\mathcal{Q}_{\ell,1,n}(\sigma)$ is the quotient of $\mathcal{H}_n^{\Bbbk}(\underline{s})$ by the cell-ideal labelled by multipartitions with more than 1 column in some component. Thus the decomposition matrix of $\mathcal{Q}_{\ell,1,n}(\sigma)$ appears as the submatrix of that of $\mathcal{H}_n^{\Bbbk}(\underline{s})$ labelled by pairs $\lambda, \mu \in \mathscr{P}_n^\ell(1)$. By Theorem 9.1, the decomposition matrix of $\mathcal{H}_n^{\Bbbk}(\underline{s})$ appears as a submatrix of that of $\mathbb{A}_n^{\Bbbk}(\sigma)$. The entries of this submatrix of the decomposition matrix of $\mathbb{A}_n^{\Bbbk}(\sigma)$ were shown to be equal to $n_{\lambda\mu}(t)$ in [BCS17, Theorem 3.16]. □

Appendix A. The many versions of this paper

This paper has gone through many arXiv iterations, through which we have developed the combinatorics and diagrammatics. The main results throughout versions 1 to 4 were Theorems A and C; we added Theorem B from version 5 onwards. Throughout versions 1 to 5, the diagrammatics and combinatorics remained very similar: we followed Webster's conventions from [Web17] where ghost strands are drawn on the left and we encoded the "charged" information via a weighting $\vartheta \in \mathbb{R}^\ell$ and a separate e-charge in $(\mathbb{Z}/e\mathbb{Z})^\ell$.

The most significant changes in the presentation of this work came in version 6 of the arXiv paper. The central idea was to dispense with Webster's weighting $\vartheta \in \mathbb{R}^\ell$ in favour of the integral lifts (of e-charges in $(\mathbb{Z}/e\mathbb{Z})^\ell$ to charges in $\mathbb{Z}^\ell$) used here, this allows us to transfer between the combinatorics of Webster, Lusztig, and Uglov seamlessly. In order to match-up this combinatorics with the diagrammatics, we had to reflect the earlier diagrams through the vertical axis (drawing ghost strands to the right) and hence obtained the diagrams which we work with in this paper (which is now available as version 7 on the arXiv). This final big change in the diagrammatics and combinatorics was inspired by the desire for an elementary proof of Theorem B and prompted by conversations with Nicolas Jacon and Maria Chlouveraki. In version 6 we introduced the idea of discretisation and we strengthened many of our intermediary results, hence simplifying the presentation of Subsection 6.2 — we were inspired by (and mimicked) the presentation of similar material from [BKW11] (for well-separated charges).

In version 6 of the paper, we chose to restrict to the reduced diagrams as our generating set for the algebra. This was in order to highlight the fact that the algebra is finitely generated, but this came at the cost of having to prove associativity (which then follows from Proposition 6.4). In this paper, we have instead chosen to include all diagrams in the generating set and then deduce that the algebra is finitely generated as a corollary of Proposition 6.4.

In this final version, we also add a new Section 8 in which we introduce the algebras $\mathbb{B}_n^{\Bbbk}(\sigma, (q : Q_0, \dots, Q_{\ell-1})$ in order to explicitly match-up the cell-modules with irreducible modules in the semisimple case.

Acknowledgements. This paper owes a special thanks to Maria Chlouveraki, Joe Chuang, Jun Hu, Nicolas Jacon, Sinead Lyle, Andrew Mathas, Liron Speyer, and Ben Webster for formative and interesting conversations, and for teaching me a great deal about cyclotomic Hecke and Cherednik algebras. The anonymous referee went above and beyond with their detailed mathematical feedback and suggestions for important improvements to earlier versions of this paper; I am thankful for their patience and hard work. This research was funded by EPSRC fellowship grant EP/V00090X/1.

References

[AJ10] S. Ariki and N. Jacon, *Dipper-James-Murphy's conjecture for Hecke algebras of type B_n*, Representation theory of algebraic groups and quantum groups, Progr. Math., vol. 284, Birkhäuser/Springer, New York, 2010, pp. 17–32. MR 2761946

[AJS94] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p*, Astérisque (1994), no. 220, 321.

[AK94] S. Ariki and K. Koike, *A Hecke algebra of $(\mathbf{Z}/r\mathbf{Z})\wr\mathfrak{S}_n$ and construction of its irreducible representations*, Adv. Math. **106** (1994), no. 2, 216–243. MR 1279219

[AM00] S. Ariki and A. Mathas, *The number of simple modules of the Hecke algebras of type $G(r,1,n)$*, Math. Z. **233** (2000), no. 3, 601–623.

[Ari95] S. Ariki, *Representation theory of a Hecke algebra of $G(r,p,n)$*, J. Algebra **177** (1995), no. 1, 164–185. MR 1356366

[Ari96] ———, *On the decomposition numbers of the Hecke algebra of $G(m,1,n)$*, J. Math. Kyoto Univ. **36** (1996), no. 4, 789–808.

[Ari01] ———, *On the classification of simple modules for cyclotomic Hecke algebras of type $G(m,1,n)$ and Kleshchev multipartitions*, Osaka J. Math. **38** (2001), no. 4, 827–837.

[Ari02] ———, *Representations of quantum algebras and combinatorics of Young tableaux*, University Lecture Series, vol. 26, American Mathematical Society, Providence, RI, 2002, Translated from the 2000 Japanese edition and revised by the author.

[Ari06] ———, *Proof of the modular branching rule for cyclotomic Hecke algebras*, J. Algebra **306** (2006), no. 1, 290–300.

[BC18] C. Bowman and A. G. Cox, *Modular decomposition numbers of cyclotomic Hecke and diagrammatic Cherednik algebras: A path theoretic approach*, Forum Math. Sigma **6** (2018).

[BCH] C. Bowman, A. Cox, and A. Hazi, *Path isomorphisms between quiver Hecke and diagrammatic Bott–Samelson endomorphism algebras*, arXiv:2005.02825, preprint.

[BCS17] C. Bowman, A. Cox, and L. Speyer, *A family of graded decomposition numbers for diagrammatic Cherednik algebras*, Int. Math. Res. Not. IMRN (2017), no. 9, 2686–2734.

[BGIL10] C. Bonnafé, M. Geck, L. Iancu, and T. Lam, *On domino insertion and Kazhdan-Lusztig cells in type B_n*, Representation theory of algebraic groups and quantum groups, Progr. Math., vol. 284, Birkhäuser/Springer, New York, 2010, pp. 33–54.

[BGS96] A. Beilinson, V. Ginzburg, and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[BI03] C. Bonnafé and L. Iancu, *Left cells in type B_n with unequal parameters*, Represent. Theory **7** (2003), 587–609.

[BJ09] C. Bonnafé and N. Jacon, *Cellular structures on Hecke algebras of type B*, J. Algebra **321** (2009), no. 11, 3089–3111.

[BK09a] J. Brundan and A. Kleshchev, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, Invent. Math. **178** (2009), no. 3, 451–484.

[BK09b] J. Brundan and A. Kleshchev, *Graded decomposition numbers for cyclotomic Hecke algebras*, Adv. Math. **222** (2009), 1883–1942.

[BKW11] J. Brundan, A. Kleshchev, and W. Wang, *Graded Specht modules*, J. Reine Angew. Math. **655** (2011), 61–87.

[BMR98] M. Broué, G. Malle, and R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127–190.

[BNS] C. Bowman, E. Norton, and J. Simental, *Characteristic-free bases and BGG resolutions of unitary simple modules for quiver Hecke and Cherednik algebras*, arXiv:1803.08736 .

[BR12] C. Bonnafé and R. Rouquier, *Cellules de Calogero–Moser*, arXiv:1302.2720v1, 2012.

[BR13] C. Bonnafé and R. Rouquier, *Calogero-Moser versus Kazhdan-Lusztig cells*, Pacific J. Math. **261** (2013), no. 1, 45–51.

[CGG12] M. Chlouveraki, I. Gordon, and S. Griffeth, *Cell modules and canonical basic sets for Hecke algebras from Cherednik algebras*, New trends in noncommutative algebra, Contemp. Math., vol. 562, Amer. Math. Soc., Providence, RI, 2012, pp. 77–89.

[CJ11] M. Chlouveraki and N. Jacon, *Schur elements and basic sets for cyclotomic Hecke algebras*, J. Algebra Appl. **10** (2011), no. 5, 979–993.

[CJ12] ———, *Schur elements for the Ariki-Koike algebra and applications*, J. Algebraic Combin. **35** (2012), no. 2, 291–311.

[CJ16] ———, *On quantized decomposition maps for graded algebras*, Algebr. Represent. Theory **19** (2016), no. 1, 135–146.

[DJM95] R. Dipper, G. James, and E. Murphy, *Hecke algebras of type B_n at roots of unity*, Proc. London Math. Soc. (3) **70** (1995), no. 3, 505–528. MR 1317512

[DJM98] R. Dipper, G. D. James, and A. Mathas, *Cyclotomic q-Schur algebras*, Math. Z. **229** (1998), no. 3, 385–416.

[EL] B. Elias and I. Losev, *Modular representation theory in type a via soergel bimodules*, arXiv:1701.00560.

[Erd97] K. Erdmann, *Representations of $\mathrm{GL}_n(K)$ and symmetric groups*, Representation theory of finite groups (Columbus, OH, 1995), Ohio State Univ. Math. Res. Inst. Publ., vol. 6, de Gruyter, Berlin, 1997, pp. 67–84.

[EW14] B. Elias and G. Williamson, *The Hodge theory of Soergel bimodules*, Ann. of Math. (2) **180** (2014), no. 3, 1089–1136.

[FLOTW99] O. Foda, B. Leclerc, M. Okado, J.-Y. Thibon, and T. Welsh, *Branching functions of $A^{(1)}_{n-1}$ and Jantzen–Seitz problem for Ariki–Koike algebras*, Adv. Math. **141** (1999), no. 2, 322–365.

[FS16] M. Fayers and L. Speyer, *Generalised column removal for graded homomorphisms between Specht modules*, J. Algebraic Combin. **44** (2016), no. 2, 393–432.

[Gec98] M. Geck, *Kazhdan-Lusztig cells and decomposition numbers*, Represent. Theory **2** (1998), 264–277.

[Gec07a] ———, *Hecke algebras of finite type are cellular*, Invent. Math. **169** (2007), no. 3, 501–517.

[Gec07b] ———, *Modular representations of Hecke algebras*, Group representation theory, EPFL Press, Lausanne, 2007, pp. 301–353.

[GGOR03] V. Ginzburg, N. Guay, E. Opdam, and R. Rouquier, *On the category $\mathcal{O}$ for rational Cherednik algebras*, Invent. Math. **154** (2003), no. 3, 617–651.

[GI13] M. Geck and L. Iancu, *Ordering Lusztig's families in type B_n*, J. Algebraic Combin. **38** (2013), no. 2, 457–489.

[GIP08] M. Geck, L. Iancu, and C. Pallikaros, *Specht modules and Kazhdan-Lusztig cells in type B_n*, J. Pure Appl. Algebra **212** (2008), no. 6, 1310–1320.

[GJ06] M. Geck and N. Jacon, *Canonical basic sets in type B_n*, J. Algebra **306** (2006), no. 1, 104–127.

[GJ11] ———, *Representations of Hecke algebras at roots of unity*, Algebra and Applications, vol. 15, Springer-Verlag London, Ltd., London, 2011.

[GM09] I. G. Gordon and M. Martino, *Calogero-Moser space, restricted rational Cherednik algebras and two-sided cells*, Math. Res. Lett. **16** (2009), no. 2, 255–262.
[GR97] M. Geck and R. Rouquier, *Centers and simple modules for Iwahori-Hecke algebras*, Finite reductive groups (Luminy, 1994), Progr. Math., vol. 141, Birkhäuser Boston, Boston, MA, 1997, pp. 251–272.
[GR01] ———, *Filtrations on projective modules for Iwahori-Hecke algebras*, Modular representation theory of finite groups (Charlottesville, VA, 1998), de Gruyter, Berlin, 2001, pp. 211–221.
[Gre07] J. A. Green, *Polynomial representations of* GL_n, augmented ed., Lecture Notes in Mathematics, vol. 830, Springer, Berlin, 2007, With an appendix on Schensted correspondence and Littelmann paths by K. Erdmann, Green and M. Schocker.
[Här99] M. Härterich, *Murphy bases of generalized Temperley-Lieb algebras*, Arch. Math. (Basel) **72** (1999), no. 5, 337–345.
[HM10] J. Hu and A. Mathas, *Graded cellular bases for the cyclotomic Khovanov–Lauda–Rouquier algebras of type A*, Adv. Math. **225** (2010), 598–642.
[HM12] J. Hu and A. Mathas, *Graded induction for Specht modules*, Int. Math. Res. Not. IMRN (2012), no. 6, 1230–1263.
[HMR] J. Hu, A. Mathas, and S. Rostam, *Skew cellularity of the Hecke algebras of type* $G(\ell,p,n)$, arXiv:2106.11459.
[Jac04] N. Jacon, *On the parametrization of the simple modules for Ariki-Koike algebras at roots of unity*, J. Math. Kyoto Univ. **44** (2004), no. 4, 729–767. MR 2118039
[Jac05] ———, *Canonical basic sets for Hecke algebras*, Infinite-dimensional aspects of representation theory and applications, Contemp. Math., vol. 392, Amer. Math. Soc., Providence, RI, 2005, pp. 33–41.
[Jac07] ———, *Crystal graphs of higher level q-deformed Fock spaces, Lusztig a-values and Ariki-Koike algebras*, Algebr. Represent. Theory **10** (2007), no. 6, 565–591.
[Jac11] ———, *Constructible representations and basic sets in type B*, J. Algebra **336** (2011), 67–83.
[JL10] N. Jacon and C. Lecouvey, *Crystal isomorphisms for irreducible highest weight* $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$*-modules of higher level*, Algebr. Represent. Theory **13** (2010), no. 4, 467–489. MR 2660857
[JM00] G. James and A. Mathas, *The Jantzen sum formula for cyclotomic q-Schur algebras*, Trans. Amer. Math. Soc. **352** (2000), no. 11, 5381–5404.
[KL79] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184.
[KL09] M. Khovanov and A. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory **13** (2009), 309–347.
[Ker] A. Kerschl, *On simple modules of cyclotomic quiver Hecke algebras of type A*, arXiv:1911.05375 , preprint.
[KMR12] A. S. Kleshchev, A. Mathas, and A. Ram, *Universal graded Specht modules for cyclotomic Hecke algebras*, Proc. Lond. Math. Soc. (3) **105** (2012), no. 6, 1245–1289.
[LM07] S. Lyle and A. Mathas, *Blocks of cyclotomic Hecke algebras*, Adv. Math. **216** (2007), no. 2, 854–878.
[LM10] ———, *Carter-Payne homomorphisms and Jantzen filtrations*, J. Algebraic Combin. **32** (2010), no. 3, 417–457.
[LM14] ———, *Cyclotomic Carter-Payne homomorphisms*, Represent. Theory **18** (2014), 117–154.
[Los16] I. Losev, *Proof of Varagnolo–Vasserot conjecture on cyclotomic categories* $\mathcal{O}$, Selecta Math. **22** (2016), no. 2, 631–668.
[LP] N. Libedinsky and D. Plaza, *Blob algebra approach to modular representation theory*, arXiv:1801.07200.
[Lus83] G. Lusztig, *Left cells in Weyl groups*, Lie group representations, I (College Park, Md., 1982/1983), Lecture Notes in Math., vol. 1024, Springer, Berlin, 1983, pp. 99–111.
[Lus03] ———, *Hecke algebras with unequal parameters*, CRM Monograph Series, vol. 18, American Mathematical Society, Providence, RI, 2003.
[Mat18] A. Mathas, *Restricting Specht modules of cyclotomic Hecke algebras*, Sci. China Math. **61** (2018), no. 2, 299–310.
[MS94] P. Martin and H. Saleur, *The blob algebra and the periodic Temperley–Lieb algebra*, Lett. Math. Phys. **30** (1994), no. 3, 189–206.
[MW03] P. Martin and D. Woodcock, *Generalized blob algebras and alcove geometry*, LMS J. Comput. Math. **6** (2003), 249–296.
[Rou08] R. Rouquier, *2-Kac–Moody algebras*, arXiv:0812.5023v1, 2008.
[RSVV16] R. Rouquier, P. Shan, M. Varagnolo, and E. Vasserot, *Categorifications and cyclotomic rational double affine Hecke algebras*, Invent. Math. **204** (2016), no. 3, 671–786.
[RW] S. Riche and G. Williamson, *Tilting modules and the p-canonical basis*, arXiv:1512.08296.
[ST54] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274–304.
[Ugl00] D. Uglov, *Canonical bases of higher-level q-deformed Fock spaces and Kazhdan–Lusztig polynomials*, Physical combinatorics (Kyoto, 1999), Progr. Math., vol. 191, Birkhuser Boston, 2000, pp. 249–299.
[Web20] B. Webster, *On graded presentations of Hecke algebras and their generalizations*, Algebr. Comb. 3 (2020), no. 1, 1–38.
[Web17] B. Webster, *Rouquier's conjecture and diagrammatic algebra*, Forum Math. Sigma **5** (2017), e27, 71.

E-mail address: chris.bowman-scargill@york.ac.uk

Department of Mathematics, University of York, Heslington, York, YO10 5DD, UK